# Quantum Diffusion of the Random Schrödinger Evolution in the Scaling Limit II. The Recollision Diagrams

# Quantum diffusion of the random Schrödinger evolution in the scaling limit II. The recollision diagrams.

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**Abstract**

We consider random Schrödinger equations on $\mathbb{R}^d$ for $d \geq 3$ with a homogeneous Anderson-Poisson type random potential. Denote by $\lambda$ the coupling constant and $\psi_t$ the solution with initial data $\psi_0$. The space and time variables scale as $x \sim \lambda^{-2-\kappa/2}, t \sim \lambda^{-2-\kappa}$ with $0 < \kappa < \kappa_0(d)$. We prove that, in the limit $\lambda \to 0$, the expectation of the Wigner distribution of $\psi_t$ converges weakly to the solution of a heat equation in the space variable $x$ for arbitrary $L^2$ initial data. The proof is based on a rigorous analysis of Feynman diagrams. In the companion paper [10] the analysis of the non-repetition diagrams was presented. In this paper we complete the proof by estimating the recollision diagrams and showing that the main terms, i.e. the ladder diagrams with renormalized propagator, converge to the heat equation.

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# 1 Introduction

The Schrödinger equation is time reversible and has no dissipation. The long time dynamics of a quantum particle in a small random environment nevertheless exhibits a stochastic behavior that can be described by a diffusion equation. Our goal is to establish this fact rigorously in the weak coupling regime. We have announced in [10] that under a scaling of space and time with inverse powers of the coupling constant $\lambda$, the Wigner distribution of the solution to the random Schrödinger equation converges to the solution of a heat equation as $\lambda \to 0$. Our approach is based on graphical expansion methods coupled with a certain truncation scheme. The first part of the proof was given in [10]; the current paper contains the second and final part. To help the orientation of the reader, we now summarize the important notations and the main result below.

The quantum dynamics of a single particle in a random potential is given by the Schrödinger equation

$$i\partial_t \psi_t = H\psi_t, \qquad \psi_t \in L^2(\mathbb{R}^d), \;\; t \in \mathbb{R} \, . \tag{1.1}$$

The Hamiltonian is a Schrödinger operator,

$$H := -\frac{1}{2}\Delta + \lambda V \, , \tag{1.2}$$

acting on $L^2(\mathbb{R}^d)$ with a random potential $V = V_\omega(x)$ and a small positive coupling constant $\lambda$. The potential is given by

$$V_\omega(x) := \int_{\mathbb{R}^d} B(x-y) \mathrm{d}\mu_\omega(y) \, , \tag{1.3}$$

where $B$ is a single site potential profile and $\mu_\omega$ is a Poisson point process on $\mathbb{R}^d$ with homogeneous unit density and with independent, identically distributed random coupling constants. More precisely, for almost all realizations $\omega$ consists of a countable, locally finite collection of points, $\{y_\gamma(\omega) \; : \; \gamma = 1, 2, \ldots\}$, and random charges $\{v_\gamma(\omega) \; : \; \gamma = 1, 2, \ldots\}$ such that the random measure is given by

$$\mu_\omega = \sum_{\gamma=1}^{\infty} v_\gamma(\omega) \delta_{y_\gamma(\omega)} \tag{1.4}$$

where $\delta_y$ denotes the Dirac mass at $y \in \mathbb{R}^d$. The Poisson process $\{y_\gamma(\omega)\}$ is independent of the charges $\{v_\gamma(\omega)\}$. The charges are real i.i.d. random variables with distribution $\mathbf{P}_v$ and with moments $m_k := \mathbf{E}_v \, v_\gamma^k$ satisfying

$$m_2 = 1, \;\; m_{2d} < \infty, \quad m_1 = m_3 = m_5 = 0 \, . \tag{1.5}$$

The expectation with respect to the random process $\{y_\gamma, v_\gamma\}$ is denoted by $\mathbf{E}$. For the single-site potential, we assume that $B$ is a spherically symmetric Schwarz function with 0 in the support of its Fourier transform.

A quantum wave $\psi \in L^2(\mathbb{R}^d)$ function can be represented on the phase space by its Wigner transform

$$W_\psi(x,v) := \int e^{2\pi i v\cdot\eta}\overline{\psi(x+\frac{\eta}{2})}\psi(x-\frac{\eta}{2})\mathrm{d}\eta\ , \tag{1.6}$$

with the convention that integrals without explicit domains denote integration over $\mathbb{R}^d$ with respect to the Lebesgue measure. Define the rescaled Wigner distribution as

$$W_\psi^\varepsilon(X,V) := \varepsilon^{-d} W_\psi\Big(\frac{X}{\varepsilon}, V\Big)\ . \tag{1.7}$$

The Fourier transform of the kinetic energy (dispersion relation) is $e(p) := \frac{1}{2}p^2$, the velocity is given by $\frac{1}{2\pi}\nabla e(p) = \frac{1}{2\pi}p$.

For any function $h : \mathbb{R}^d \to \mathbf{C}$ and energy value $e \geq 0$ we introduce the notation

$$[h](e) := \int h(v)\delta(e-e(v))\mathrm{d}v := \int_{\Sigma_e} h(q)\ \frac{\mathrm{d}\nu(q)}{|\nabla e(q)|}\ , \tag{1.8}$$

where $\mathrm{d}\nu(q)$ is the restriction of the Lebesgue measure onto the energy surface $\Sigma_e := \{q\ :\ e(q) = e\}$ that is the ball of radius $\sqrt{2e}$. The normalization of the measure $[\cdot]_e$ is given by

$$[1](e) := c_{d-1}(2e)^{\frac{d}{2}-1}\ , \tag{1.9}$$

where $c_{d-1}$ is the volume of the unit sphere $S^{d-1}$.

We consider the scaling

$$t = \lambda^{-\kappa}\big(\lambda^{-2}T\big), \quad x = \lambda^{-\kappa/2}\big(\lambda^{-2}X\big) = X/\varepsilon, \quad \varepsilon = \lambda^{\kappa/2+2} \tag{1.10}$$

with some $\kappa \geq 0$. On the kinetic scale, $\kappa = 0$, the limiting equation is the linear Boltzmann equation with a collision kernel

$$\sigma(u,v) := 2\pi|\widehat{B}(u-v)|^2\delta(e(u)-e(v))\ . \tag{1.11}$$

The generator of the corresponding momentum jump process $v(t)$ on the energy surface $\Sigma_e$, $e > 0$, is

$$L_e f(v) := \int \mathrm{d}u\ \sigma(u,v)[f(u)-f(v)], \qquad e(v) = e\ . \tag{1.12}$$

The choice $\kappa > 0$ corresponds to the long time limit of the Boltzmann equation with diffusive scaling. The Markov process $\{v(t)\}_{t\geq 0}$ with generator $L_e$ is exponentially mixing (see Lemma A.1 in the Appendix). Let

$$D_{ij}(e) := \frac{1}{(2\pi)^2}\int_0^\infty \mathcal{E}_e\big[v^{(i)}(t)v^{(j)}(0)\big]\mathrm{d}t\ , \qquad v = (v^{(1)},\dots,v^{(d)}), \quad i,j = 1,2,\dots d,$$

be the velocity autocorrelation matrix, where $\mathcal{E}_e$ denotes the expectation with respect to this Markov process in equilibrium. By the spherical symmetry of $\widehat{B}$ and $e(U)$, the autocorrelation matrix is constant times the identity:

$$D_{ij}(e) = D_e\, \delta_{ij}, \qquad D_e := \frac{1}{(2\pi)^2 d} \int_0^\infty \mathcal{E}_e\big[v(t)\cdot v(0)\big]\,\mathrm{d}t\,. \tag{1.13}$$

The main result is the following theorem.

**Theorem 1.1** *Let $d \geq 3$ and $\psi_0 \in L^2(\mathbb{R}^d)$ be a normalized initial wave function. Let $\psi(t) := \exp(-itH)\psi_0$ solve the Schrödinger equation (1.1). Let $\mathcal{O}(x,v)$ be a Schwarz function on $\mathbb{R}^d \times \mathbb{R}^d$. For any $e > 0$, let $f$ be the solution to the heat equation*

$$\partial_T f(T,X,e) = D_e\, \Delta_X f(T,X,e) \tag{1.14}$$

*with the initial condition*

$$f(0,X,e) := \delta(X)\Big[|\widehat{\psi}_0(v)|^2\Big](e)\,.$$

*Then there exist $0 < \kappa_0(d) \leq 2$ such that for $0 < \kappa < \kappa_0(d)$ and for $\varepsilon$ and $\lambda$ related by* (1.10)*, the rescaled Wigner distribution satisfies*

$$\lim_{\lambda\to 0} \int \mathrm{d}X \int \mathrm{d}v\, \mathcal{O}(X,v)\mathbf{E}W^{\varepsilon}_{\psi(\lambda^{-\kappa-2}T)}(X,v) = \int \mathrm{d}X \int \mathrm{d}v\, \mathcal{O}(X,v) f(T,X,e(v))\,, \tag{1.15}$$

*and the limit is uniform on $T \in [0, T_0]$ with any fixed $T_0$. In $d = 3$ one can choose* $\kappa_0(3) = 1/500$.

Our result shows that the quantum dynamics on the time scale $t \sim \lambda^{-2-\kappa}$ is given by a heat equation and the diffusion coefficient can be computed from the long time behavior of the underlying Boltzmann dynamics. Heuristically, this statement can be understood from two facts. First, the Boltzmann equation correctly describes the limit of the quantum evolution on the kinetic time scale $t \sim \lambda^{-2}$, $\lambda \to 0$ (see [18], [7], [5] and [15]). Second, the long time limit of the linear Boltzmann evolution is a diffusion. This two step limiting argument is, however, misleading (e.g. in $d = 2$, localization is expected to occur at all values of $\lambda$, so that no diffusion occurs). In higher dimensions, quantum correlations that are small on the kinetic scale and are neglected in the first limit may become important on the longer time scale, too. To prove Theorem 1.1, we need to control the full quantum dynamics up to the time scale $t \sim \lambda^{-2-\kappa}$ and prove that the quantum correlations are not sufficiently strong to destroy the heuristic picture.

The approach of this paper applies also to lattice models and yields a derivation of Brownian motion from the Anderson model [8, 9]. The dynamics of the Anderson model was postulated by Anderson [3] to be localized for large coupling constant $\lambda$ and extended for small coupling constant (away from the band edges and in dimension $d \geq 3$). The localization conjecture was first established rigorously by Goldsheid, Molchanov and Pastur [13] in one dimension, by Fröhlich-Spencer [12], and later by Aizenman-Molchanov [1] in several dimensions, and many other works have since contributed to this field.

The progress for the extended state regime, however, has been limited. It was proved by Klein [14] that all eigenfuctions are extended on the Bethe lattice (see also [2, 11]). In Euclidean space, Schlag, Shubin and Wolff [17] proved that the eigenfunctions cannot be localized in a region smaller than $\lambda^{-2+\delta}$ for some $\delta > 0$ in $d = 2$. Chen [5], extended this result to all dimensions $d \geq 2$ and $\delta = 0$ (with logarithmic corrections). Extended states for Schrödinger equation with a sufficiently decaying random potential were proven by Rodnianski and Schlag [16] and Bourgain [4] (see also [6]). However, all known results for Anderson-type models in Euclidean space are in regions where the dynamics have typically finitely many effective collisions. Under the diffusive scaling of this paper, see (1.10), the number of effective scatterings is a negative fractional power of the scaling parameter. In particular, it goes to infinity in the scaling limit, as it should be the case if we aim to obtain a Brownian motion.

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# 2 Summary of Part I

## 2.1 Notations

We introduce a few notations. The letters $x, y, z$ will denote configuration space variables, while $p, q, r, u, v, w$ will be used for $d$-dimensional momentum variables. The norm $\|\cdot\|$ denotes the standard $L^2(\mathbb{R}^d)$ norm and we set

$$\|f\|_{m,n} := \sum_{|\alpha|\leq n} \|\langle x\rangle^m \partial^\alpha f(x)\|_\infty$$

with $\langle x\rangle := (2+x^2)^{1/2}$ (here $\alpha$ is a multiindex). The bracket $(\cdot\,,\cdot)$ denotes the standard scalar product on $L^2(\mathbb{R}^d)$ and $\langle\cdot\,,\cdot\rangle$ will denote the pairing between the Schwarz space and its dual on the phase space $\mathbb{R}^d \times \mathbb{R}^d$. The Fourier transform is denoted by hat. For functions on the phase space, $f(x,v)$, $x, v \in \mathbb{R}^d$, Fourier transform will always be understood in the first variable only:

$$\widehat{f}(\xi, v) := \int e^{-2\pi i\xi\cdot x} f(x,v)\mathrm{d}v \; .$$

By the regularization argument in Section 3.2 of [10], we can cutoff the high momentum regime of $\psi_0$ and we can multiply $B$ by a $\lambda$-dependent cutoff function in momentum space at a negligible error. When dealing with estimates for any fixed $\lambda > 0$, we can thus assume that

$$\operatorname{supp} \widehat{\psi}_0 \ \text{ is compact}, \qquad \operatorname{supp} \widehat{B}(p) \subset \{|p| \le \lambda^{-\delta}\} \tag{2.1}$$

for any fixed $\delta > 0$. Universal constants and constants that depend only on the dimension $d$, on the final time $T_0$ and on $\psi_0$ and $B$ will be denoted by $C$. The same applies to the hidden constants in the $O(\cdot)$ and $o(\cdot)$ notations.

In [10] the self-energy operator was defined as the multiplication operator in momentum space

$$\theta(p) := \Theta(e(p)), \qquad \Theta(\alpha) := \lim_{\varepsilon \to 0+} \Theta_\varepsilon(\alpha) \ , \qquad \Theta_\varepsilon(\alpha) := \Theta_\varepsilon(\alpha, r) \tag{2.2}$$

for any $r$ with $e(r) = \alpha$, where

$$\Theta_\varepsilon(\alpha, r) := \int \frac{|\widehat{B}(q-r)|^2 \mathrm{d}q}{\alpha - e(q) + i\varepsilon} \ . \tag{2.3}$$

The function $\Theta(\alpha)$ is Hölder continuous with exponent $\frac{1}{2}$, and it decays as $\langle \alpha \rangle^{-1/2}$ (Lemma 3.1 and 3.2 in [10]). If we write $\Theta(\alpha) = \mathcal{R}(\alpha) - i\mathcal{I}(\alpha)$, where $\mathcal{R}(\alpha)$ and $\mathcal{I}(\alpha)$ are real functions, and recall $Im(x + i0)^{-1} = -\pi\delta(x)$, we have

$$\mathcal{I}(\alpha) = -\operatorname{Im} \Theta(\alpha) = \pi \int \delta(e(q) - \alpha) |\widehat{B}(q-r)|^2 \mathrm{d}q \tag{2.4}$$

for any $r$ satisfying $\alpha = e(r)$.

The dispersion relation was renormalized by adding the self-energy term:

$$\omega(p) := e(p) + \lambda^2 \theta(p) \ ,$$

and the Hamiltonian was rewritten as

$$H = H_0 + \widetilde{V}, \qquad H_0 := \omega(p), \qquad \widetilde{V} := \lambda V - \lambda^2 \theta(p) \ . \tag{2.5}$$

The renormalization compensates for the immediate recollisions in the Duhamel expansion (see Section 3). The rate of the immediate recollisions is of order $\lambda^2$, thus their total effect is $\lambda^2 t \gg 1$. The renormalization removes this instability. We note that $\omega(p)$ is not the self-consistently renormalized dispersion relation, but only its approximation up to $O(\lambda^4)$. Since this error is negligible on our time scale due to $\lambda^4 t \ll 1$, we use $\omega$ to simplify the technicalities associated with the analysis of the self-consistent dispersion relation.

We define the renormalized propagator (with $\eta$-regularization):

$$R_\eta(\alpha, v) := \frac{1}{\alpha - \omega(v) + i\eta} \, .$$

In Appendix B.1 we prove the following estimates on the renormalized propagator.

**Lemma 2.1** *Suppose that $\lambda^2 \geq \eta \geq \lambda^{2+4\kappa}$ with $\kappa \leq 1/12$. Then we have,*

$$\int \frac{|h(p-q)| \mathrm{d}p}{|\alpha - \omega(p) + i\eta|} \leq \frac{C\|h\|_{2d,0} \, |\log \lambda| \, \log\langle\alpha\rangle}{\langle\alpha\rangle^{1/2} \langle |q| - \sqrt{2|\alpha|}\rangle} \, , \tag{2.6}$$

*and for $0 \leq a < 1$*

$$\int \frac{|h(p-q)| \mathrm{d}p}{|\alpha - \omega(p) + i\eta|^{2-a}} \leq \frac{C_a \|h\|_{2d,0} \, \lambda^{-2(1-a)}}{\langle\alpha\rangle^{a/2} \langle |q| - \sqrt{2|\alpha|}\rangle} \, , \tag{2.7}$$

$$\int \frac{|h(p-q)| \mathrm{d}p}{|\alpha - e(p) + i\eta|^{2-a}} \leq \frac{C_a \|h\|_{2d,0} \, \eta^{-2(1-a)}}{\langle\alpha\rangle^{a/2} \langle |q| - \sqrt{2|\alpha|}\rangle} \, . \tag{2.8}$$

*For $a = 0$ and with $h := \widehat{B}$, the following more precise estimate holds. There exists a constant $C_0$, depending only on finitely many $\|B\|_{k,k}$ norms, such that*

$$\int \frac{\lambda^2 |\widehat{B}(p-q)|^2 \, \mathrm{d}p}{|\alpha - \overline{\omega}(p) - i\eta|^2} \leq 1 + C_0 \lambda^{-12\kappa} \big[\lambda + |\alpha - \omega(q)|^{1/2}\big] \, . \tag{2.9}$$

## 2.2 The expansion

We expand the unitary kernel of $H = H_0 + \widetilde{V}$ (see (2.5)) by the Duhamel formula and after taking the expectation, we organize the expansion into sums of Feynman diagrams. In order to avoid the infinite summations (1.4) in the expansion, we have reduced the problem to a large finite box, $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$ with periodic boundary conditions (see Section 3.3 of [10]). The infinite volume Poisson process $\mu_\omega$ was replaced with

$$\mu'_\omega = \sum_{\gamma=1}^{M} v'_\gamma \delta_{y'_\gamma}$$

where $M$ is a Poisson random variable with mean $|\Lambda_L|$, the points $\{y'_\gamma\}_{\gamma=1}^M$ are uniformly distributed on $\Lambda_L$ and the real charges $v'_\gamma$ have distribution $\mathbf{P}_v$. All random variables are independent. Lemma 3.4 of [10] guarantees that these modifications have no effect on the final result if $L \to \infty$ is taken before any other limit.

After the Duhamel expansion, taking the expectation and letting $L \to \infty$, we regain the infinite volume formulas for the Feynman graphs. In [10] we used primes to denote the restricted quantities, but to avoid the heavy notation here we will omit them, except when stating the theorems. All quantities throughout this section are understood on $\Lambda_L$ with $M$ random points.

We recall the Duhamel formula from Section 4 of [10]. For any fixed integer $N \geq 1$

$$\psi_t := e^{-itH}\psi_0 = \sum_{n=0}^{N-1} \psi_n(t) + \Psi_N(t) \; , \tag{2.10}$$

with

$$\psi_n(t) := (-i)^n \int_0^t [\mathrm{d}s_j]_1^{n+1} \;\; e^{-is_{n+1}H_0} \widetilde{V} e^{-is_n H_0} \widetilde{V} \ldots \widetilde{V} e^{-is_1 H_0} \psi_0 \tag{2.11}$$

$$\Psi_N(t) := (-i) \int_0^t \mathrm{d}s \, e^{-i(t-s)H} \widetilde{V} \psi_{N-1}(s) \tag{2.12}$$

with the notation

$$\int_0^t [\mathrm{d}s_j]_1^n := \int_0^t \ldots \int_0^t \Big( \prod_{j=1}^n \mathrm{d}s_j \Big) \delta\Big( t - \sum_{j=1}^n s_j \Big) \; .$$

Substituting $\widetilde{V} = -\lambda^2\theta(p) + \sum_{\gamma=1}^M \lambda V_\gamma$ with $V_\gamma(x) := v_\gamma B(x - y_\gamma)$, the terms in (2.11) and (2.12) are summations over collision history. Denote by $\tilde{\Gamma}_n$, $n \leq \infty$, the set of sequences

$$\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n), \qquad \tilde{\gamma}_j \in \{1, 2, \ldots, M\} \cup \{\vartheta\} \tag{2.13}$$

and by $W_{\tilde{\gamma}}$ the associated potential

$$W_{\tilde{\gamma}} := \begin{cases} \lambda V_{\tilde{\gamma}} & \text{if} \quad \tilde{\gamma} \in \{1, \ldots, M\} \\ -\lambda^2 \theta(p) & \text{if} \quad \tilde{\gamma} = \vartheta \; . \end{cases}$$

The tilde refers to the fact that the additional $\{\vartheta\}$ symbol is also allowed. An element $\tilde{\gamma}_j$ of the sequence $\tilde{\gamma} \in \tilde{\Gamma}$ is called *potential label* and $j$ is called *potential index* if $\tilde{\gamma}_j \in \{1, 2, \ldots M\}$, otherwise they are called *$\vartheta$-label* and *$\vartheta$-index*, respectively. A potential label carries a factor $\lambda$, a $\vartheta$-label carries $\lambda^2$.

For any $\tilde{\gamma} \in \tilde{\Gamma}_n$ we define the following fully expanded wave function with truncation

$$\psi_{*t,\tilde{\gamma}} := (-i)^{n-1} \int_0^t [\mathrm{d}s_j]_1^n \;\; W_{\tilde{\gamma}_n} e^{-is_n H_0} W_{\tilde{\gamma}_{n-1}} \ldots e^{-is_2 H_0} W_{\tilde{\gamma}_1} e^{-is_1 H_0} \psi_0 \tag{2.14}$$

and without truncation

$$\psi_{t,\tilde{\gamma}} := (-i)^n \int_0^t [\mathrm{d}s_j]_1^{n+1} \;\; e^{-is_{n+1}H_0} W_{\tilde{\gamma}_n} e^{-is_n H_0} W_{\tilde{\gamma}_{n-1}} \ldots e^{-is_2 H_0} W_{\tilde{\gamma}_1} e^{-is_1 H_0} \psi_0 \; . \tag{2.15}$$

In the notation the star $(*)$ will always refer to truncated functions. Each term $\psi_{t,\tilde\gamma}$ along the expansion procedure is characterized by its order $n$ and by a sequence $\tilde\gamma \in \tilde\Gamma_n$. The main terms are given by *non-repetitive* sequences that contain only potential labels, i.e. we define

$$\Gamma_k^{nr} := \Big\{ \gamma = (\gamma_1, \ldots, \gamma_k) \; : \; \gamma_j \in \{1, \ldots, M\}, \; \gamma_i \neq \gamma_j \text{ if } i \neq j \Big\} \subset \tilde\Gamma_n \, . \tag{2.16}$$

The sum of the corresponding elementary wave functions is denoted by

$$\psi_{t,k}^{nr} := \sum_{\gamma \in \Gamma_k^{nr}} \psi_{t,\gamma} \, . \tag{2.17}$$

The rate of collisions is $O(\lambda^2)$, thus the total number of collisions is typically of order $k \sim \lambda^2 t$. We thus set

$$K := [\lambda^{-\delta}(\lambda^2 t)] \tag{2.18}$$

($[\,\cdot\,]$ denotes integer part) to be an upper threshold for the number of collisions in the expanded terms. Here $\delta = \delta(\kappa) > 0$ is a small positive number to be fixed later on.

## 2.3 Structure of the proof.

In Section 5 of [10] the Main Theorem was proved from three key theorems. For completeness, we repeat these three statements here. Recall that the prime indicates restriction to $\Lambda_L$ and hence dependence on $L, M$

**Theorem 2.2 ($L^2$-estimate of the error terms)** *Let $t = O(\lambda^{-2-\kappa})$ and $K$ given by (2.18). If $\kappa < \kappa_0(d)$ and $\delta$ is sufficiently small (depending only on $\kappa$), then*

$$\lim_{\lambda \to 0} \lim_{L \to \infty} \mathbf{E}' \Big\| \psi_t' - \sum_{k=0}^{K-1} \psi_{t,k}'^{\,nr} \Big\|_L^2 = 0 \, .$$

*In $d = 3$ dimensions, one can choose $\kappa_0(3) = \frac{1}{500}$.*

**Theorem 2.3 (Only the ladder diagram contributes)** *Let $\kappa < \frac{2}{34d+39}$, $\varepsilon = \lambda^{2+\kappa/2}$, $\eta = \lambda^{2+\kappa}$, $t = O(\lambda^{-2-\kappa})$, and $K$ given by (2.18). For a sufficiently small positive $\delta$ and for any $1 \le k \le K$ we have*

$$\lim_{L \to \infty} \mathbf{E}' \|\psi_{t,k}'^{\,nr}\|_L^2 = V_\lambda(t,k) + O\Big(\lambda^{\frac{1}{3} - (\frac{17}{3}d + \frac{13}{2})\kappa - O(\delta)}\Big) \tag{2.19}$$

$$\lim_{L \to \infty} \langle \widehat{\mathcal{O}}_L, \mathbf{E}' \widehat{W}^{\varepsilon}_{\psi_{t,k}'^{\,nr}} \rangle_L = W_\lambda(t,k,\mathcal{O}) + O\Big(\lambda^{\frac{1}{3} - (\frac{17}{3}d + \frac{13}{2})\kappa - O(\delta)}\Big) \tag{2.20}$$

*as $\lambda \ll 1$. Here*

$$V_\lambda(t,k) := \frac{\lambda^{2k} e^{2t\eta}}{(2\pi)^2} \iint_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\beta \, e^{i(\alpha-\beta)t} \int \Big( \prod_{j=1}^{k+1} \mathrm{d}p_j \Big) \, |\widehat{\psi}_0(p_1)|^2$$
$$\times \prod_{j=1}^{k+1} \overline{R_\eta(\alpha,p_j)} R_\eta(\beta,p_j) \prod_{j=1}^{k} |\widehat{B}(p_{j+1}-p_j)|^2 \tag{2.21}$$

$$W_\lambda(t,k,\mathcal{O}) := \frac{\lambda^{2k} e^{2t\eta}}{(2\pi)^2} \iint_{-\infty}^{\infty} \mathrm{d}\alpha \mathrm{d}\beta \, e^{i(\alpha-\beta)t} \int \mathrm{d}\xi \int \Big( \prod_{j=1}^{k+1} \mathrm{d}v_j \Big) \widehat{\mathcal{O}}(\xi, v_{k+1}) \overline{\widehat{W}^{\varepsilon}_{\psi_0}}(\xi, v_1)$$
$$\times \prod_{j=1}^{k+1} \overline{R_\eta\Big(\alpha, v_j + \frac{\varepsilon\xi}{2}\Big)} R_\eta\Big(\beta, v_j - \frac{\varepsilon\xi}{2}\Big) \prod_{j=1}^{k} |\widehat{B}(v_j - v_{j+1})|^2 \, . \tag{2.22}$$

The definition (2.22) does not apply literally to the free evolution term $k=0$; this term is defined separately:

$$W_\lambda(t, k=0, \mathcal{O}) := \int \mathrm{d}\xi \mathrm{d}v \, e^{it\varepsilon v\cdot\xi} \, e^{2t\lambda^2 \operatorname{Im}\theta(v)} \, \widehat{\mathcal{O}}(\xi, v) \overline{\widehat{W_0}}(\varepsilon\xi, v) \, . \tag{2.23}$$

**Theorem 2.4 (The ladder diagram converges to the heat equation)** *Under the conditions of Theorem 2.3 and setting $t = \lambda^{-2-\kappa}T$, we have*

$$\lim_{\lambda\to 0} \sum_{k=0}^{K-1} W_\lambda(t,k,\mathcal{O}) = \int \mathrm{d}X \int \mathrm{d}v \, \mathcal{O}(X,v) f(T, X, e(v)) \, , \tag{2.24}$$

*where $f$ is the solution to the heat equation* (1.14).

The main result of [10] was the proof of Theorem 2.3. In Sections 3–5 of this paper we prove Theorem 2.2 and in Section 6 we prove Theorem 2.4. We now explain the ideas behind these theorems. The actual estimates are somewhat weaker than the heuristics predicts.

The proof of Theorem 2.2 consists in controlling the wave functions of collision histories that contain $\vartheta$-labels or repeated potential labels. The repeated potential labels correspond to recollisions with the same obstacle. Immediate recollision with the same obstacle occurs with an amplitude $O(\lambda^2)$. Over the total history of the evolution, this would yield a large contribution of order $\lambda^2 t \gg 1$. However, the wave functions of these collision histories will be resummed with those containing $\vartheta$-labels. Thanks to the choice of the renormalization counterterm, $\lambda^2\theta(p)$, the contributions of the immediate recollisions and the $\vartheta$-labels cancel each other up to leading order. Each resummed term thus has an amplitude $O(\lambda^4 t)$. The full propagator in the error term, $\Psi_N(t)$ (2.12), however, will be estimated by unitarity (4.4). This estimate effectively loses an extra $t$ factor. To compensate for it, we have to continue the expansion up to two immediate recollisions in the error term.

The amplitudes of the non-immediate recollisions are much smaller and they can be estimated individually. Heuristically, the probability of such recollisions can be understood in classical mechanics. Since the mean free path is $\lambda^{-2}$, returning to an already visited obstacle after visiting another obstacle at distance $\lambda^{-2}$ has probability $O(\lambda^{2d})$. Another scenario is when the particle collides with obstacle $\gamma_1$, then it bounces back from a nearby obstacle $\gamma_2$, $|\gamma_1 - \gamma_2| = O(1)$, and then it recollides with $\gamma_1$. This situation is atypical and it is penalized by $O(\lambda^4)$ because the time elapsed between these collisions is $O(1)$ while the collision cross-section is $O(\lambda^2)$. In conclusion, the probability of a non-immediate recollision among the total $k \sim \lambda^2 t$ collisions is at most $O(k\lambda^4)$, thus the total effect of these recollisions on our time scale is negligible, even when multiplied with the additional factor $t$ from the unitarity estimates for $\Psi_N(t)$.

This outline neglects the key analytic difficulty originated from the growth of the combinatorics of Feynman diagrams. The amplitude of non-repetition wave functions can be written as a sum of $k!$ Feynman diagrams. Only one of them, the ladder diagram, contributes to the heat equation. All other diagrams can be estimated by $O(\lambda^2)$ due to phase cancellations. This estimate is not sufficient to sum up all diagrams since their number, $k! \sim \exp(\lambda^{-const})$, is exponentially large. The size of a few combinatorially simple diagrams is indeed $O(\lambda^2)$, but much stronger estimates were obtained in [10] as the combinatorial complexity of the diagram increases. This improvement balances the increased combinatorial factor for more complicated diagrams and it allows us to control the expansion for non-repetition wave functions for time scale $t \sim \lambda^{-2-\kappa}$.

In this paper, we extend the classification scheme to include *all* diagrams arising from the Duhamel expansion. Thanks to the stopping rules, this part involves only a few extra collisions. The main idea of this paper is to design a surgery of Feynman diagrams so that a general diagram can be decomposed into a repetition and a non-repetition part: the repetition part involves only a few variables and the integration can be estimated accurately; the non-repetition part is reduced to Theorem 2.3. The errors from the surgery are controlled by the small factors from the repetition part. This renders all repetition diagrams negligible. Thus we prove that among all diagrams only the ladder diagrams without repetition contribute to the final heat equation. In practice, the scheme used in this paper is much more complicated than is stated here. But this description gives a good first idea.

Finally, the proof of Theorem 2.4 is a fairly explicit but delicate calculation involving singular integrals. The proof shows how the long time evolution of the Boltzmann

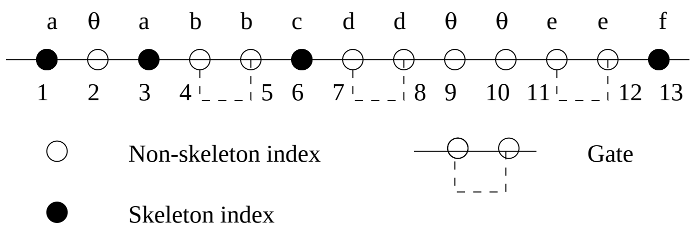


Figure 1: Gates and skeletons

equation emerges from the ladder diagrams.

# 3 The stopping rules

We use the Duhamel expansion to identify the non-repetition error terms to be estimated in Theorem 2.2. This method allows for the flexibility that at every new term of the expansion we perform the separation into elementary waves, $\psi_{*s,\widetilde{\gamma}}$, and we can decide whether we want to stop (keeping the full propagator as in (2.12)) or we continue to expand that term further. This decision will depend on the collision history, $\widetilde{\gamma}$, and it will be given by a precise algorithm, the stopping rules. The key idea is that once the collision history $\tilde{\gamma}$ is "sufficiently" atypical, i.e. it contains either atypical recollision or too many collisions, we stop the expansion for that elementary wave function immediately to reduce the number and the complexity of the expanded terms.

Not every recollision is atypical. An immediate second collision with the same obstacle contributes to the main term; this is actually the reason why the dispersion relation had to be corrected with the self-energy $\lambda^2\theta(p)$.

In a sequence $\tilde{\gamma}$ we thus identify the *immediate recollisions* inductively starting from $\tilde{\gamma}_1$ (due to their graphical picture, they are also called *gates*). The gates must involve potential labels and not $\vartheta$. For example, the sequence $\tilde{\gamma} = (a, \vartheta, a, b, b, c, d, d, \vartheta, \vartheta, e, e, f)$ has three gates (see Fig. 1). In the sequence $(a, b, b, c, c, c)$ there are two gates. Any potential label which does not belong to a gate will be called *skeleton label.* The index $j$ of a skeleton label $\gamma_j$ in $\tilde{\gamma}$ is called *skeleton index.* The set of skeleton indices is $S(\tilde{\gamma})$. Similar terminology is used for the gates. In the first example $1, 3, 6, 13$ are skeleton indices and $a, a, c, f$ are skeleton labels, in the second example $1, 6$ are skeleton indices and $a, c$ are skeleton labels. The $\vartheta$ terms are never part of the skeleton.
This definition is recursive so we can identify skeleton indices successively along the expansion procedure. Notice that a skeleton index may become a gate index at a later stage of the expansion, but never the other way around.

The formal definition is as follows. Let $I_n := \{1, 2, \dots, n\}$.

**Definition 3.1 (Skeleton labels and indices)** *Let $\tilde\gamma = (\tilde\gamma_1, \tilde\gamma_2, \dots, \tilde\gamma_n) \in \tilde\Gamma_n$ and let $\tilde\gamma^* := (\tilde\gamma_1, \tilde\gamma_2, \dots, \tilde\gamma_{n-1})$ be its truncation. The set of skeleton indices of $\tilde\gamma$, $S(\tilde\gamma) \subset I_n$, is defined inductively (on the length of $\tilde\gamma$) as follows. If $\tilde\gamma \in \tilde\Gamma_1$, then $S(\tilde\gamma) := \{1\}$ if $\tilde\gamma_1 \neq \vartheta$ and $S(\tilde\gamma) := \emptyset$ otherwise. Furthermore, for any $\tilde\gamma \in \tilde\Gamma_n$, $n \geq 2$, let*

$$S(\tilde\gamma) := \begin{cases} S(\tilde\gamma^*) & \text{if} \quad \tilde\gamma_n = \vartheta \\ S(\tilde\gamma^*) \setminus \{n-1\} & \text{if} \quad n-1 \in S(\tilde\gamma^*) \quad \text{and} \quad \tilde\gamma_n = \tilde\gamma_{n-1} \\ S(\tilde\gamma^*) \cup \{n\} & \text{if} \quad \tilde\gamma_n \neq \vartheta \quad \text{and} \quad [\tilde\gamma_n \neq \tilde\gamma_{n-1} \quad \text{or} \quad n-1 \notin S(\tilde\gamma^*)] \, . \end{cases}$$

*Finally, $\gamma_n$ is called skeleton label if $n \in S(\gamma)$.*

For any $\tilde\gamma \in \tilde\Gamma_n$, let

$$k(\tilde\gamma) := |S(\tilde\gamma)|,$$

be the number of skeleton indices in $\tilde\gamma$, let

$$I_n^\theta(\tilde\gamma) := \{ j \; : \; \tilde\gamma_j = \vartheta \}$$

be the set of $\theta$-indices and $t(\tilde\gamma) := |I_n^\theta(\tilde\gamma)|$. Denote the total $\lambda^2$-power collected from non-skeleton indices by

$$r(\tilde\gamma) := \frac{1}{2}[n - k(\tilde\gamma)] + t(\tilde\gamma) \tag{3.25}$$

Notice that $r(\tilde\gamma)$ is integer.

The exact stopping rule requires somewhat tedious definitions of different types of elementary wave functions. First we give these definitions intuitively, state our final representation formula for $\psi_t$ using these concepts, then we give the precise definitions and prove the formula.

Sequences where the only repetitions in potential labels occur within the gates are called *non-repetitive* sequences. A special case is the set of non-repetitive sequences, $\Gamma_k^{nr}$, without gates and $\theta$-labels. The repetitive sequences are divided into the following categories (Fig. 2). If two non-neighboring skeleton labels coincide, then the collision history includes a *genuine (non-immediate) recollision*. If a skeleton label coincides with a gate label, then we have a *triple collision* of the same obstacle. If two neighboring skeleton labels coincide and are not immediate recollisions because there are gates or $\vartheta$'s in between, then we have a *nest.*

We stop the expansion at an elementary truncated wave function (2.14) characterized by $\tilde\gamma$, if any of the following happens (precise definitions will be given in Definition 3.3).

• The number of skeleton indices in $\tilde\gamma$ reaches $K$ (see (2.18)). We denote the sum of the truncated elementary non-repetitive wave functions up to time $s$ with at most one $\lambda^2$ power from the non-skeleton labels or $\vartheta$'s and with $K$ skeleton indices by $\psi_{*s,K}^{(\leq 1),nr}$. The superscript $(\leq 1)$ refers the number of collected $\lambda^2$ powers from non-skeleton labels.

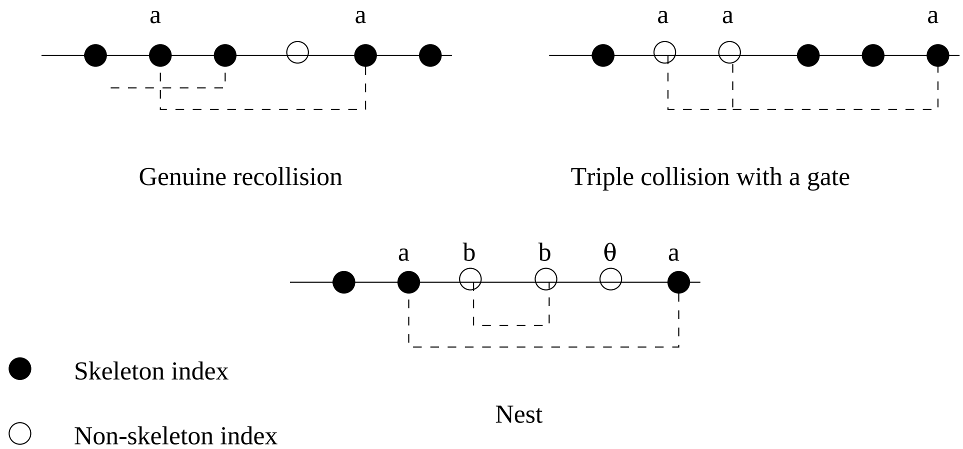


Figure 2: Repetition patterns

• We have collected $\lambda^4$ from non-skeleton labels. We denoted by $\psi_{*s,k}^{(2),last}$ the sum of the truncated elementary wave functions up to time $s$ with two $\lambda^2$ power from the non-skeleton indices (the word *last* indicates that the last $\lambda$ power was collected at the last collision).

• We observe a repeated skeleton label, i.e., a recollision or a nest. The corresponding wave functions are denoted by $\psi_{*s,k}^{(\leq 1),rec}, \psi_{*s,k}^{(\leq 1),nest}$.

• We observe three identical potential labels, i.e., a triple collision. The corresponding wave functions are denoted by $\psi_{*s,k}^{(\leq 1),tri}$.

Finally, $\psi_{t,k}^{(\leq 1),nr}$ denotes the sum of non-repetitive elementary wave functions without truncation (i.e. up to time $t$) with at most one $\lambda^2$ power from the non-skeleton indices or $\vartheta$'s and with $k$ skeleton indices. In particular, the non-repetition wave functions without gates and $\vartheta$'s (denoted by $\psi_{t,k}^{nr}$ above) contribute to this sum. For notational consistence, we will rename $\psi_{t,k}^{(0),nr} := \psi_{t,k}^{nr}$ to explicitly indicate the number of $\lambda^2$-powers collected from gates or $\vartheta$'s.

This stopping rule gives rise to the following representation.

**Proposition 3.2** *[Duhamel formula] For any $K \geq 1$ we have*

$$\psi_t = e^{-itH}\psi_0 = \sum_{k=0}^{K-1} \psi_{t,k}^{(\leq 1),nr} \tag{3.26}$$

$$-i\int_0^t \mathrm{d}s\; e^{-i(t-s)H}\left\{\psi_{*s,K}^{(\leq 1),nr}+\sum_{k=0}^{K}\left(\psi_{*s,k}^{(2),last}+\psi_{*s,k}^{(\leq 1),rec}+\psi_{*s,k}^{(1),nest}+\psi_{*s,k}^{(1),tri}\right)\right\}.$$

*Proof of Proposition 3.2.* We start with the precise definitions. For $\tilde{\gamma}\in\tilde{\Gamma}_n$ and $\ell<n$ we introduce the notation $\tilde{\gamma}_{[1,\ell]}:=(\tilde{\gamma}_1,\ldots,\tilde{\gamma}_\ell)$ to denote the beginning segment, or truncation, of length $\ell$ of the sequence $\tilde{\gamma}$.

**Definition 3.3 (Sets of sequences)** *For $0\leq r\leq 2$, $k\leq n$ we let*

$$\tilde{\Gamma}_{n,k}^{(r)}:=\{\tilde{\gamma}\in\tilde{\Gamma}_n\;:\;k(\tilde{\gamma})=k,\;r(\tilde{\gamma})=r\}$$

*be the set of sequences with $k$ skeleton indices and $\lambda^{2r}$ collected from non-skeleton indices. Let*

$$\tilde{\Gamma}_{n,k}^{(r),nr}:=\{\tilde{\gamma}\in\tilde{\Gamma}_{n,k}^{(r)}\;:\;[\tilde{\gamma}_j=\tilde{\gamma}_{j'}\neq\theta]\Longrightarrow[|j-j'|=1,j,j'\notin S(\tilde{\gamma})]\}$$

*be the set of* **non-repetitive** *sequences. For $r=0$ we have $n=k$ and we set $\Gamma_k^{nr}:=\tilde{\Gamma}_{k,k}^{(0),nr}$. Let*

$$\tilde{\Gamma}_{n,k}^{(r),last}:=\{\tilde{\gamma}\in\tilde{\Gamma}_{n,k}^{(r),nr}\;:\;\tilde{\gamma}_n\notin S(\tilde{\gamma})\}$$

*be the set of non-repetitive sequences whose last element is non-skeleton. Let*

$$\tilde{\Gamma}_n^{nr}:=\bigcup_{k\leq n}\bigcup_{r\leq 2}\tilde{\Gamma}_{n,k}^{(r),nr}$$

*be the set of all non-repetitive sequences of length $n$ and let*

$$\tilde{\Gamma}_n^{*}:=\{\tilde{\gamma}\in\tilde{\Gamma}_n\setminus\tilde{\Gamma}_n^{nr}\;:\;\tilde{\gamma}_{[1,n-1]}\in\tilde{\Gamma}_{n-1}^{nr}\}$$

*be the set of sequences that are repetitive, but their proper truncations are non-repetitive. We let*

$$\tilde{\Gamma}_{n,k}^{(r),tri}:=\tilde{\Gamma}_n^{*}\cap\{\tilde{\gamma}\in\tilde{\Gamma}_{n,k}^{(r)}\;:\;\tilde{\gamma}_n\in S(\tilde{\gamma}),\;\exists j\leq n-2,\tilde{\gamma}_j,\tilde{\gamma}_{j+1}\notin S(\tilde{\gamma}),\tilde{\gamma}_n=\tilde{\gamma}_j=\tilde{\gamma}_{j+1}\}$$

*be the set of* **triple-collision sequences***, i.e. sequences whose last entry $\tilde{\gamma}_n$ is a part of a triple collision with a gate. Let*

$$\tilde{\Gamma}_{n,k}^{(r),rec}:=\tilde{\Gamma}_n^{*}\cap\{\tilde{\gamma}\in\tilde{\Gamma}_{n,k}^{(r)}\setminus\tilde{\Gamma}_{n,k}^{(r),tri}\;:\;\exists j\in S(\tilde{\gamma}),j\leq n-2,\tilde{\gamma}_j=\tilde{\gamma}_n,S(\tilde{\gamma})\cap\{j+1,\ldots,n-1\}\neq\emptyset\}$$

*be the set of* **recollision sequences***. Finally, let*

$$\tilde{\Gamma}_{n,k}^{(r),nest}:=\tilde{\Gamma}_n^{*}\cap\{\tilde{\gamma}\in\tilde{\Gamma}_{n,k}^{(r)}\setminus(\tilde{\Gamma}_{n,k}^{(r),rec}\cup\tilde{\Gamma}_{n,k}^{(r),tri})\;:\;\exists j\in S(\tilde{\gamma}),j\leq n-2,\tilde{\gamma}_j=\tilde{\gamma}_n\}$$

*be the set of* **nested sequences***. In all cases we introduce the notation*

$$\tilde{\Gamma}_{n,k}^{(\leq R),\#}:=\bigcup_{r=0}^{R}\tilde{\Gamma}_{n,k}^{(r),\#}$$

*where $\#=tri,rec,nest,nr,last$ refers to the* **structure** *of the wave function.*

Notice that the last index $n$ of any $\tilde{\gamma} \in \tilde{\Gamma}^*_n$ is a skeleton index, in particular $\tilde{\Gamma}^{(r),last}_{n,k} \cap \tilde{\Gamma}^*_n = \emptyset$. This index can create a repetition in three different ways: triple collision, recollision or nest. It is therefore clear from the definition, that the sets $\tilde{\Gamma}^{(r),tri}_{n,k}$, $\tilde{\Gamma}^{(r),rec}_{n,k}$, $\tilde{\Gamma}^{(r),nest}_{n,k}$ and $\tilde{\Gamma}^{(r),nr}_{n,k}$ for $0 \leq r \leq 2$ are disjoint. Moreover, for triple collision and nested sequences we have $r \geq 1$. The next lemma shows that these sets include the appropriate beginning segment of any infinite sequence.

**Lemma 3.4** *Let a positive integer $K$ be given. Let $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots) \in \tilde{\Gamma}_\infty$ be an infinite sequence. Then there exist a unique $k \leq K$ and $n \in [k, k+4]$ such that the truncation of length $n$ of $\tilde{\gamma}$, $\tilde{\gamma}_{[1,n]}$ belongs to the (disjoint) union*

$$\tilde{\Gamma}(n, K) := \tilde{\Gamma}^{(\leq 1),nr}_{n,K} \cup \bigcup_{k=0}^{K} \left( \tilde{\Gamma}^{(1),tri}_{n,k} \cup \tilde{\Gamma}^{(\leq 1),rec}_{n,k} \cup \tilde{\Gamma}^{(1),nest}_{n,k} \cup \tilde{\Gamma}^{(2),last}_{n,k} \right). \tag{3.27}$$

*Proof.* We look at the increasing family of truncated sequences $\tilde{\gamma}_{[1,2]}$, $\tilde{\gamma}_{[1,3]}, \ldots$ inductively. If $\tilde{\gamma}_{[1,n]} \in \tilde{\Gamma}^{(r),nr}_{n,K}$ for some $n$ and $r \leq 1$, then it falls into the first set of (3.27).

Otherwise there is an $n \leq K+4$ such that $\tilde{\gamma}_{[1,n-1]}$ is non-repetitive with $r \leq 1$, but $\tilde{\gamma}_{[1,n]}$ is either repetitive or $r(\tilde{\gamma}_{[1,n]}) = 2$. If it is repetitive, then $\tilde{\gamma}_n$ is a skeleton index, so $r(\tilde{\gamma}_{[1,n]}) = r(\tilde{\gamma}_{[1,n-1]}) \leq 1$, and the repetition can be a triple collision, recollision or nest with $k \leq K$. If $\tilde{\gamma}_{[1,n]}$ is non-repetitive, then the $r$ has increased from $r(\tilde{\gamma}_{[1,n-1]}) = 1$ to $r(\tilde{\gamma}_{[1,n]}) = 2$ and $\tilde{\gamma}_n$ is non-skeleton. These four possibilities correspond to the remaining sets in (3.27). The disjointness of these sets follows from their definition. $\square$

For $0 \leq r \leq 2$ and $\# = rec, nest, tri, last, nr$, let

$$\psi^{(r),\#}_{(*)t,k} := \sum_{n=k+r}^{k+2r} \sum_{\tilde{\gamma} \in \tilde{\Gamma}^{(r),\#}_{n,k}} \psi_{(*)t,\tilde{\gamma}}$$

be the wave function with $k$ skeleton labels, with $\lambda^{2r}$ total power collected from non-skeleton terms and with recollision, nest, triple collision or no repetition (with the last collision being skeleton or not) specified by $\#$. The notation $(*)$ indicates that the same definition is used for the wave functions with or without truncation. Finally we set

$$\psi^{(\leq 1),\#}_{(*)t,k} := \psi^{(0),\#}_{(*)t,k} + \psi^{(1),\#}_{(*)t,k}.$$

We stop the expansion at the elementary truncated wave function (2.14) characterized by $\tilde{\gamma} \in \tilde{\Gamma}_n$, if $\tilde{\gamma}$ falls into one of the sets in (3.27), but none of its proper truncations $\tilde{\gamma}_{[1,n']}$ fell into the appropriate sets (3.27) with $n$ replaced with $n'$. Lemma 3.4 shows that the expansion is stopped for every term for a unique reason. This procedure proves Proposition 3.2. $\square$

# 4 Error terms

The content of Theorem 2.2 is that the main contribution to the wave function $\psi_t$ in (3.26) comes from the fully expanded non-recollision terms with $r = 0$, i.e $\psi_{t,k}^{(0),nr}$. Here we show that the contribution of all other terms are negligible. Each error term in (3.26) has a specific reason to be small. The result can be summarized in the following Theorem which is proven in Sections 4 and 5. We recall that prime indicates restriction to $\Lambda_L$.

**Theorem 4.1** *We assume $t = \lambda^{-2-\kappa}T$, $T \in [0, T_0]$, and $1 \le k \le K$. If $\kappa < \frac{2}{34d+39}$, then*

$$\lim_{L\to\infty} \mathbf{E}' \|\psi'^{(r),\#}_{*t,k}\|^2 = o(\lambda^{4+2\kappa+2\delta}), \qquad \lambda \to 0, \tag{4.1}$$

*for the following choices of parameters: $\{\# = rec,\ r = 0, 1\}$, $\{\# = nest, tri,\ r = 1\}$ or $\{\# = last,\ r = 2\}$. Furthermore, for $k = K$ and $r = 0, 1$,*

$$\lim_{L\to\infty} \mathbf{E}' \|\psi'^{(r),nr}_{*t,K}\|^2 = o(\lambda^{4+2\kappa+2\delta}) \ , \tag{4.2}$$

*and for $k < K$,*

$$\lim_{L\to\infty} \mathbf{E}' \|\psi'^{(1),nr}_{t,k}\|^2 = o(\lambda^{2\kappa+2\delta}) \ . \tag{4.3}$$

*Proof of Theorem 2.2 using Theorem 4.1.* By (3.26) and the unitarity of the operator $e^{-i(t-s)H}$, we have

$$\mathbf{E}' \Big\| \int_0^t \mathrm{d}s\, e^{-i(t-s)H} \sum_{k\le K} \psi'^{(r),\#}_{*s,k} \Big\|^2 \le t^2 K \sum_{k=0}^{K} \sup_{0\le s\le t} \mathbf{E}' \Big\| \psi'^{(r),\#}_{*s,k} \Big\|^2 \ , \tag{4.4}$$

where the value of $r$ is determined by $\#$ according to the terms on the right hand side of (3.26). The non-repetition terms with $r = 1$ (first term on the right hand side of (3.26)) are fully expanded and there is no need for unitarity. After a Schwarz inequality,

$$\mathbf{E}' \Big\| \sum_{k=0}^{K-1} \psi'^{(1),nr}_{t,k} \Big\|^2 \le K \sum_{k=0}^{K-1} \mathbf{E}' \|\psi'^{(1),nr}_{t,k}\|^2.$$

Given Theorem 4.1, all these error terms are negligible if we first take $L \to \infty$ and then $\lambda \to 0$. $\square$

The natural way to prove Theorem 4.1 would be to rewrite $\mathbf{E}\|\psi\|^2$ into a sum of Feynman graphs, similarly to Proposition 7.2 in [10], then to identify a repetition subgraph of a few vertices (recollisions, nests etc.) that renders the graphs small and remove them by graph surgeries after extracting a small factor. We then should sum up the remaining graphs on the core indices for all possible permutations and lumps as in Section

9 of [10]. For the sake of technical simplifications, however, at the price of a smaller $\kappa$ we follow a somewhat different path. We first identify the vertices in the graph (called *core indices*) that carry no complication whatsoever (no repetition, no gate, no $\theta$). We then symmetrize the non-core indices in $\psi$ and $\bar{\psi}$, by a Schwarz inequality to reduce the number of repetition patterns. For graphs with sufficient high combinatorial complexity (with large joint degree, see Definition 4.4 below) we simply neglect the possible gain from the repetition patterns by removing them from the graph with a crude estimate. The necessary small factor will come from Proposition 9.2 of [10] with $q$ being large. For graphs with low complexity, the gain comes from analyzing the repetition patterns case by case. Since there are not too many low-complexity graphs, we can neglect the possible gain from the complexity and still sum up for all combinatorial patterns of the core indices.

## 4.1 Feynman graphs and their values

In this section we collect the necessary definitions from [10] to estimate the values of Feynman graphs. More details can be found in Sections 7.1 and 7.2 of [10].

The Feynman graph is an oriented circle graph on $N \geq 2$ vertices and with two distinguished vertices, denoted by $0$, $0^*$. The number of vertices between $0$ and $0^*$ are $n$ and $n'$, in particular $N = n + n' + 2$. The vertex set can thus be identified with the set $\mathcal{V} = \mathcal{V}_{n,n'} := \{0, 1, 2, \ldots, n, 0^*, \tilde{n}', \widetilde{n'-1}, \ldots, \tilde{1}\}$ equipped with the circular ordering. We set $I_n := \{1, 2, \ldots n\}$ and $\widetilde{I}_{n'} := \{\tilde{1}, \tilde{2}, \ldots, \tilde{n}'\}$. The set of oriented edges, $\mathcal{L}(\mathcal{V})$, can be partitioned into $\mathcal{L}(\mathcal{V}) = \mathcal{L} \cup \widetilde{\mathcal{L}}$ so that $\mathcal{L}$ contains the edges between $I_n \cup \{0, 0^*\}$ and $\widetilde{\mathcal{L}}$ contains the edges between $\widetilde{I}_{n'} \cup \{0, 0^*\}$.

For $v \in \mathcal{V}$ we use the notation $v - 1$ and $v + 1$ for the vertex right before and after $v$ in the circular ordering. We also denote $e_{v-} = (v-1, v)$ and $e_{v+} = (v, v+1)$ the edge right before and after the vertex $v$, respectively. For each $e \in \mathcal{L}(\mathcal{V})$, we introduce a momentum $w_e$ and a real number $\alpha_e$ associated to this edge. The collection of all momenta is denoted by $\mathbf{w} = \{w_e \ : \ e \in \mathcal{L}(\mathcal{V})\}$ and $\mathrm{d}\mathbf{w} = \otimes_e \mathrm{d}w_e$ is the Lebesgue measure. The notation $v \sim e$ will indicate that an edge $e$ is adjacent to a vertex $v$.

Let $\mathbf{P} = \{P_\mu \ : \ \mu \in I\}$ be a partition of the set $\mathcal{V} \setminus \{0, 0^*\} = I_n \cup \widetilde{I}_{n'}$ into nonempty, pairwise disjoint sets, where $I = I(\mathbf{P})$ is the index set to label the sets in the partition. Let $m(\mathbf{P}) := |I(\mathbf{P})|$. The sets $P_\mu$ are called $\mathbf{P}$-*lumps* or just *lumps*. We assign an auxiliary variable, $u_\mu \in \mathbb{R}^d$, $\mu \in I(\mathbf{P})$, to each lump. The vector of auxiliary momenta is denoted by $\mathbf{u} := \{u_\mu \ : \ \mu \in I(\mathbf{P})\}$. We will always assume that they add up to zero

$$\sum_{\mu \in I(\mathbf{P})} u_\mu = 0 \tag{4.5}$$

and that they satisfy $|u_\mu| \leq O(\lambda^{-2\kappa - 4\delta})$. The set of all partitions of the vertex set $\mathcal{V} \setminus \{0, 0^*\}$ is denoted by $\mathcal{P}_{\mathcal{V}}$. When we wish to indicate the $n, n'$ dependence and identify $\mathcal{V}\setminus\{0,0^*\}$ with $I_n\cup\widetilde{I}_{n'}$, then the set of all partitions on $I_n\cup\widetilde{I}_{n'}$ will be denoted by $\mathcal{P}_{n,n'}$ instead of $\mathcal{P}_{\mathcal{V}}$.

For any $P\subset\mathcal{V}$, we let

$$L_+(P) := \{(v,v+1)\in\mathcal{L}(\mathcal{V}) \;:\; v+1\notin P,\; v\in P\}$$

denote the set of edges that go out of $P$, with respect to the orientation of the circle graph, and similarly $L_-(P)$ denote the set of edges that go into $P$. We set $L(P) := L_+(P)\cup L_-(P)$.

For any $\xi\in\mathbb{R}^d$ we define the following product of delta functions

$$\Delta(\mathbf{P},\mathbf{w},\mathbf{u}) := \delta\Big(\xi + \sum_{e\in L_\pm(\{0^*\})} \pm w_e\Big) \prod_{\mu\in I(\mathbf{P})} \delta\Big(\sum_{e\in L_\pm(P_\mu)} \pm w_e - u_\mu\Big)\,. \tag{4.6}$$

The sign $\pm$ indicates that momenta $w_e$ is added or subtracted depending whether the edge $e$ is outgoing or incoming, respectively.

For each subset $\mathcal{G}\subset\mathcal{V}\setminus\{0,0^*\}$, we define

$$\mathcal{N}_{\mathcal{G}}(\mathbf{w}) := \prod_{e\sim 0} |\widehat{\psi}_0(w_e)| \prod_{v\in\mathcal{V}\setminus\{0,0^*\}\setminus\mathcal{G}} |\widehat{B}(w_{e_{v-}} - w_{e_{v+}})| \prod_{v\in\mathcal{G}} \langle w_{e_{v-}} - w_{e_{v+}}\rangle^{-2d}\,. \tag{4.7}$$

We also define the restricted Lebesgue measure

$$\mathrm{d}\mu(w) := \mathbf{1}(|w|\le\zeta)\mathrm{d}w\,,\quad \zeta := \lambda^{-\kappa-3\delta},\qquad \mathrm{d}\mu(\mathbf{w}) := \otimes_e \mathrm{d}\mu(w_e)\,. \tag{4.8}$$

On the support of $\Delta$ this restriction will not substantially influence our integrals (see (7.9)–(7.10) of [10]).

With these notations, we define, for any $\mathbf{P}\in\mathcal{P}_{\mathcal{V}}$ and $g=0,1,2,\ldots$, the **$E$-value of the partition**

$$E_g(\mathbf{P},\mathbf{u},\boldsymbol{\alpha}) := \lambda^{N-2} \sup_{\mathcal{G}\,:\,|\mathcal{G}|\le g} \int \mathrm{d}\mu(\mathbf{w}) \prod_{e\in\mathcal{L}(\mathcal{V})} \frac{1}{|\alpha_e - \omega(w_e) + i\eta|}\; \Delta(\mathbf{P},\mathbf{w},\mathbf{u})\mathcal{N}_{\mathcal{G}}(\mathbf{w})\,. \tag{4.9}$$

The $E$-value depends also on the parameters $\lambda,\eta$, but we will not specify them in the notation. We will also need a **truncated version** of this definition:

$$E_{*g}(\mathbf{P},\mathbf{u},\boldsymbol{\alpha}) := \lambda^{N-2} \sup_{\mathcal{G}\,:\,|\mathcal{G}|\le g} \int \mathrm{d}\mu(\mathbf{w}) \prod_{\substack{e\in\mathcal{L}(\mathcal{V})\\ e\notin L(\{0^*\})}} \frac{1}{|\alpha_e - \omega(w_e) + i\eta|}\; \Delta(\mathbf{P},\mathbf{w},\mathbf{u})\mathcal{N}_{\mathcal{G}}(\mathbf{w})\,. \tag{4.10}$$

Let $\mathbf{P}\in\mathcal{P}_{n,n'}$ be a partition on the set $I_n\cup\widetilde{I}_{n'}$. The lumps of a partition containing only one vertex will be called *single lumps.* The vertices 0 and $0^*$ will not be considered single lumps. Let $G = G(\mathbf{P})$ be the set of edges that go into a single lump and let $g(\mathbf{P}) := |G(\mathbf{P})|$ be its cardinality. In case of $n = n'$, we will use the shorter notation $\mathcal{V}_n = \mathcal{V}_{n,n}$, $\mathcal{P}_n = \mathcal{P}_{n,n}$ etc. We will always have

$$|n - n'| \le g(\mathbf{P}) \le 4, \qquad n, n' \le K. \tag{4.11}$$

We also introduce a function $Q$ that will represent the momentum dependence of the observable. We can assume, for convenience, that $\|Q\|_\infty \le 1$. We define

$$\mathcal{M}(\mathbf{w}) := \prod_{e\in\mathcal{L}\cap G}[-\overline{\theta(w_e)}] \prod_{e\in\widetilde{\mathcal{L}}\cap G}[-\theta(w_e)] \prod_{\substack{e\in\mathcal{L}\setminus G\\ e\not\sim 0^*}} \overline{\widehat{B}(w_e - w_{e+1})} \prod_{\substack{e\in\widetilde{\mathcal{L}}\setminus G\\ e\not\sim 0}} \widehat{B}(w_e - w_{e+1})$$
$$\times \overline{\widehat{\psi}(w_{e_{0+}})}\widehat{\psi}(w_{e_{0-}})Q\Big[\frac{1}{2}(w_{e_{0^*-}} + w_{e_{0^*+}})\Big] \tag{4.12}$$

where $e + 1$ denotes the edge succeeding $e \in \mathcal{L}(\mathcal{V})$ in the circular ordering.

Let $\alpha, \beta \in \mathbb{R}$, $\mathbf{P} \in \mathcal{P}_{n,n'}$, and define

$$V(\mathbf{P}, \alpha, \beta) := \lambda^{n+n'+g(\mathbf{P})} \int \mathrm{d}\mu(\mathbf{w}) \prod_{e\in\mathcal{L}} \frac{1}{\alpha - \overline{\omega}(w_e) - i\eta} \prod_{e\in\widetilde{\mathcal{L}}} \frac{1}{\beta - \omega(w_e) + i\eta} \tag{4.13}$$
$$\times\Delta(\mathbf{P}, \mathbf{w}, \mathbf{u} \equiv 0)\mathcal{M}(\mathbf{w}) \; .$$

The truncated version, $V_*(\mathbf{P}, \alpha, \beta)$, is defined analogously but the $\alpha$ and $\beta$ denominators that correspond to $e \in L(\{0^*\})$ are removed.

We set $Y := \lambda^{-100}$ and define

$$V_{(*)}(\mathbf{P}) := \frac{e^{2t\eta}}{(2\pi)^2} \iint_{-Y}^{Y} \mathrm{d}\alpha \mathrm{d}\beta \; e^{it(\alpha-\beta)} V_{(*)}(\mathbf{P}, \alpha, \beta) \tag{4.14}$$

and

$$E_{(*)g}(\mathbf{P}, \mathbf{u}) := \frac{e^{2t\eta}}{(2\pi)^2} \iint_{-Y}^{Y} \mathrm{d}\alpha \mathrm{d}\beta \; E_{(*)g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha}) \; , \tag{4.15}$$

where $\boldsymbol{\alpha}$ in $E_{(*)g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha})$ is defined as $\alpha_e = \alpha$ for $e \in \mathcal{L}$ and $\alpha_e := \beta$ for $e \in \widetilde{\mathcal{L}}$. The notation $(*)$ indicates the same formulas with and without truncation. We will call these numbers the *$V$-value* and *$E$-value of the partition* $\mathbf{P}$, or sometimes, of the corresponding Feynman graph. Strictly speaking, the $V$- and $E$-values depend on $\xi$ through $\Delta$ and the $V$-value depends on the choice of $Q$ as well. When necessary, we will make these dependencies explicit in the notation, e.g. $E_\xi$ or $V_\xi(\mathbf{P}; Q)$. The $E$-value is a convenient estimate for the $V$-value of the graph (see (7.14) of [10])

$$\big|V_{(*)}(\mathbf{P})\big| \le (C\lambda)^g \; E_{(*)g}(\mathbf{P}, \mathbf{u} \equiv 0) \tag{4.16}$$

with $g = g(\mathbf{P})$. We will use the notation $E_{(*)g}(\mathbf{P}) := E_{(*)g}(\mathbf{P}, \mathbf{u} \equiv 0)$.

For the graphical representation of the Duhamel expansion we will really need

$$V^\circ_{(*)}(\mathbf{P}) := \frac{e^{2t\eta}}{(2\pi)^2} \iint_{\mathbb{R}} \mathrm{d}\alpha \mathrm{d}\beta \, V_{(*)}(\mathbf{P}, \alpha, \beta) \ , \tag{4.17}$$

i.e. a version of $V_{(*)}(\mathbf{P})$ with unrestricted $\mathrm{d}\alpha \, \mathrm{d}\beta$ integrations. (The circle superscript in $V^\circ$ will refer to the unrestricted version of $V$). The difference between the restricted and unrestricted $V$-values are negligible even when we sum them up for all partitions (Lemma 7.1 of [10]).

Sometimes we will use the numerical labelling of the edges. We will label the edge between $(j-1, j)$ by $e_j$, the edge between $(\widetilde{j}, \widetilde{j-1})$ by $e_{\widetilde{j}}$ and we set $e_{n+1} := (n, 0^*)$, $e_{\widetilde{n'+1}} := (0^*, \tilde{n}')$, $e_1 = (0,1)$ and $e_{\widetilde{1}} := (\widetilde{1}, 0)$. Therefore the edge set $\mathcal{L} = \mathcal{L}(\mathcal{V}_{n,n'})$ is identified with the index set $I_{n+1} \cup \widetilde{I}_{n'+1}$ and we set

$$p_j := w_{e_j}, \qquad \tilde{p}_j := w_{e_{\widetilde{j}}}. \tag{4.18}$$

## 4.2 Resummation for core indices

We need to identify the non-repetitive potential labels in a sequence.

**Definition 4.2 (Core of a sequence)** *Let $\tilde\gamma \in \tilde\Gamma_n$, then the set of* **core indices** *of $\tilde\gamma$ is defined as*

$$I_n^{core}(\tilde\gamma) := \Big\{ j \in S(\tilde\gamma) \ : \ \tilde\gamma_j \neq \tilde\gamma_i, \ \forall \, i \neq j \Big\}$$

*and we set $c(\tilde\gamma) = |I_n^{core}(\tilde\gamma)|$. The corresponding $\tilde\gamma_j$ labels are called* **core labels**. *The subsequence of core labels form an element in $\Gamma_c^{nr}$, i.e. a sequence of different potential labels. The elements of*

$$I_n^{nc}(\tilde\gamma) := I_n \setminus [I_n^{core}(\tilde\gamma) \cup I_n^\theta(\tilde\gamma)]$$

*are called* **non-core potential indices**.

*In other words, the core indices are those skeleton indices (Definition 3.1) that do not participate in any recollision, gate, triple collision or nest.* Given the stopping rules (Section 3), the number of non-core potential indices and $\theta$-indices together is at most 4. The number of core indices $c = c(\tilde\gamma)$ is related to the number of skeleton indices $k = k(\tilde\gamma)$ as follows

$$c := \begin{cases} k & \text{if} \quad \# = nr, last \\ k-1 & \text{if} \quad \# = triple \\ k-2 & \text{if} \quad \# = nest, rec. \end{cases} \tag{4.19}$$

For any $\tilde\gamma \in \tilde\Gamma_n$, the index set $I_n$ is partitioned as $I_n = I_n^{core} \cup I_n^{nc} \cup I_n^\theta$ into core indices, non-core potential indices and $\theta$-indices. Let $\tau = \tau(\tilde\gamma) := (\tau_1, \tau_2, \ldots, \tau_c) \in \Gamma_c^{nr}$ denote the core labels of the sequence $\tilde{\gamma}$. We also introduce the notation $\tau_{[a,b]} := (\tau_a, \tau_{a+1}, \dots, \tau_b)$ if $a \le b$, and $\tau_{[a,b]} = \emptyset$ otherwise.

Now let $\tilde{\gamma} \in \tilde{\Gamma}(n,K) \cup \tilde{\Gamma}^{(1),nr}_{n,k}$ (see (3.27) and Definition 3.3 for the notation). We recall that the total number of gates and $\theta$'s is given by $r$. The possible values of $r$ are determined by $\# = rec, nest, triple, last$ according to (3.27) or $r = 1$ if $\# = nr$ and $k < K$. We will refer to a gate or $\theta$ index as a gate/$\theta$-index in short.

We rewrite each error term in (3.26) by first summing over core labels $\tau$. For fixed number of core indices $c$ we sum over all possible locations of non-core indices. If a non-core index is inserted between the $(w-1)$-th and $w$-th core indices, we characterize its location by $w$.

The locations of non-core indices within the sequence are given by a **location code** $w$. For example, if $\# = last$, then $w \in I_{c+1}$ encodes that the first gate/$\theta$-index is located between the $(w-1)$-th and $w$-th core indices. The location of the second gate/$\theta$-index need not be encoded because it is fixed to be after the last core index. If $\# = rec$ and $r = 0$, then $w \in I_c$ encodes that the first recollision label is between the $(w-1)$-th and $w$-th core indices. The most complicated case is $\# = rec$, $r = 1$, when the code $w$ consists of two numbers, $w = (w_1, w_2) \in I_c \times I_{c+1}$, where $w_1$ and $w_2$ encode the location of the recollision and gate/$\theta$, respectively. If $w_1 = w_2$, an extra binary code determines whether the gate/$\theta$ is immediately before or after the recollision. The set of possible location codes therefore depends on $\#$, $c$ and $r$, and it will be denoted by $W = W_c^{(r),\#}$.

The detailed description of the set $W$ in the other cases is obvious but lengthy and we omit the formal details. The precise structure of $W$ is not important, but we remark that its cardinality satisfies $|W| \le (c+1)^2$ in all cases. We thus have the following resummation formula:

$$\psi'^{(r),\#}_{(*)t,k} = \sum_{\tau \in \Gamma^{nr}_c} \sum_{w \in W} \psi'^{(r),\#}_{(*)t,\tau,w} \,, \tag{4.20}$$

where $\psi'^{(r),\#}_{(*)t,\tau,w}$ is the wave function with core labels $\tau$, with structure $\#$, with $r$ gates/$\theta$-indices and with location of non-core indices given by $w$. We note that the wave function $\psi'^{(r),\#}_{(*)t,\tau,w}$ includes a summation over the non-core labels with the restriction that they are distinct from the core labels $\tau$.

Having specified the locations of the $r$ gates/$\theta$-indices within the sequence of core indices, we introduce another code $h \in \{g, \theta\}^r$, called **gate-code**, to specify whether there is a gate or a $\theta$ at the given location. This gives the decomposition

$$\psi'^{(r),\#}_{(*)t,\tau,w} = \sum_{h \in \{\theta, g\}^r} \psi'^{h,\#}_{(*)t,\tau,w} \tag{4.21}$$

with the obvious definition of $\psi'^{h,\#}_{(*)t,\tau,w}$.

## 4.3 Symmetrization of the non-core indices

Starting from (4.20), we can use the Schwarz inequality

$$\begin{aligned}\mathbf{E}'\|\psi'^{(r),\#}_{(*)t,k}\|^2 &= \sum_{w,w'\in W}\mathbf{E}'\Big\langle\sum_{\tau\in\Gamma_c^{nr}}\psi'^{(r),\#}_{(*)t,\tau,w}\,,\sum_{\tau'\in\Gamma_c^{nr}}\psi'^{(r),\#}_{(*)t,\tau',w'}\Big\rangle\\ &\le |W|\sum_{w\in W}\sum_{\tau,\tau'\in\Gamma_c^{nr}}\mathbf{E}'\Big\langle\psi'^{(r),\#}_{(*)t,\tau,w}\,,\psi'^{(r),\#}_{(*)t,\tau',w}\Big\rangle\,,\end{aligned}$$

where $c$ is given by $k$ and $\#$ according to (4.19), and recall that $W$ depends on $\#, c, r$.

Notice that any non-core potential label appears in pair (in a gate, nest or recollision) and none of them appear more than six times by the stopping rules. Since the first, third and fifth moments of the random variables $v_\gamma$ are zero, the expectation in (4.22) is nonzero only if $\tau'$ and $\tau$ are paired, i.e. if there is a permutation $\sigma\in\mathfrak{S}_c$ such that $\tau'=\sigma(\tau)$, meaning $\tau_i'=\tau_{\sigma(i)}$. Therefore

$$\begin{aligned}\mathbf{E}'\|\psi'^{(r),\#}_{(*)t,k}\|^2 &\le |W|\sum_{w\in W}\sum_{\sigma\in\mathfrak{S}_c}\sum_{\tau\in\Gamma_c^{nr}}\mathbf{E}'\Big\langle\psi'^{(r),\#}_{(*)t,\tau,w}\,,\psi'^{(r),\#}_{(*)t,\sigma(\tau),w}\Big\rangle\\ &\le |W|\sum_{w\in W}\sum_{\sigma\in\mathfrak{S}_c}\sum_{h,h'\in\{g,\theta\}^r}\sum_{\tau\in\Gamma_c^{nr}}\mathbf{E}'\Big\langle\psi'^{h,\#}_{(*)t,\tau,w}\,,\psi'^{h',\#}_{(*)t,\sigma(\tau),w}\Big\rangle\,.\end{aligned}\tag{4.22}$$

Note that each wave function $\psi'^{(r),\#}_{(*)t,\tau,w}$ has been further decomposed into a sum over $h$-codes according to (4.21). However, we did not estimate the $h\neq h'$ cross terms by an additional Schwarz inequality. The term with a gate must cancel the term with a $\theta$ exactly at the same location, i.e. $\psi^g$ and $\psi^\theta$ would not individually be negligible, but their sum is of smaller order.

Notice also that each wave function $\psi'^{h,\#}_{(*)t,\tau,w}$ may involve summations over one or two further non-core potential labels. We use the convention that the recollision or nest label is denoted by $\nu$, the label of the gate or triple is denoted by $\mu$. In case of a second gate, $\#=last$, its label will be $\mathring{\mu}$. According to the non-repetition rules, $(\nu,\mu,\mathring{\mu})$ may not coincide with each other or with any element of $\tau$. In the products $\overline{\psi}'^{h,\#}_{(*)t,\tau,w}\psi'^{h',\#}_{(*)t,\sigma(\tau),w}$ there is no repetition among $\tau$ and $\sigma(\tau)$ indices other than the ones prescribed by $\sigma$. However, if the additional non-core labels within $\psi'^{h',\#}_{(*)t,\sigma(\tau),w}$ are denoted by $\nu',\mu'$ or $\mathring{\mu}'$, then there may be a few coincidences between primed and non-primed non-core labels. Those coincidences are allowed that do not violate the non-repetition rules requiring that $\nu,\mu,\mathring{\mu}$ are distinct and their primed counterparts are also distinct among themselves.

Once the number of core indices $c$, a location-code $w$ and a gate-code $h$ are fixed, this defines a unique insertion of the non-core indices into the sequence of core indices $I_c$. The core and non-core indices in the given order can be identified with $I_n$, where $n$, the total number of collisions, is given by $n = k + r + |\{j \ : \ h_j = g\}|$ and $k$ is given by (4.19). This naturally defines an embedding map $s = s^h_w : I_c \mapsto I_n$. Similarly, $\widetilde{s}^{h'}_w : \widetilde{I}_c \mapsto \widetilde{I}_n$ can be defined. The precise definition depends on $\#$, $r$ and $h$ in a natural way. For illustration, we describe the most complicated case; $\# = rec$, $r = 1$ and $h = g$. In this case $n = c+4$. For definitiveness, we consider the case when the recollision precedes the gate $w_1 \le w_2$. In this case the complete collision sequence is $(\tau_{[1,w_1-1]}, \nu, \tau_{[w_1,w_2-1]}, \mu, \mu, \tau_{[w_2,c]}, \nu)$. Let

$$s^{h=g}_w(j) := \begin{cases} j & \text{if} \quad j < w_1 \\ j+1 & \text{if} \quad w_1 \le j < w_2 \\ j+3 & \text{if} \quad w_2 \le j \le c \ . \end{cases}$$

With this notation, the pairing of core indices, originally determined by $\sigma \in \mathfrak{S}_c$, is given by the pairs $\{s^h_w(j), \widetilde{s}^{h'}_w(\sigma(j))\}$ as subsets of the full index set $I_n \cup \widetilde{I}_{n'}$.

Given $(\#, c, \sigma, w, h, h')$, we define the partition $\mathbf{D}_0 = \mathbf{D}_0(\#, c, \sigma, w, h, h')$ of $I_n \cup \widetilde{I}_{n'}$ by lumping *only* those indices that *are required* to carry the same potential label by the prescribed structure $\#$ and the gates. The vertices with $\theta$ always remain a single lump, the remaining vertices are paired.

For example, if $\# = rec$, $r = 1$, $w = (w_1, w_2)$, $h = h' = g$, we obtain

$$\mathbf{D}_0 := \Big\{ \{s(j), \widetilde{s}(\sigma(j))\}_{j \in I_c}, \{w_1, n\}, \{w_2+1, w_2+2\}, \{\widetilde{w}_1, \widetilde{n}\}, \{\widetilde{w_2+1}, \widetilde{w_2+2}\} \Big\}$$

(see Fig. 3) and all other cases are similar. The elements of $\mathbf{D}_0$ consisting of pairs of core indices, $\{s(j), \widetilde{s}(\sigma(j))\}_{j \in I_c}$, are called *core elements* of $\mathbf{D}_0$ and those elements of $\mathbf{D}_0$ that contain non-core indices will be called *non-core elements*. In the example above, the last four elements are the non-core elements of $\mathbf{D}_0$ describing the two gates and two recollisions. The non-core potential labels $\nu, \mu, \nu', \mu'$ will correspond to these non-core elements, respectively.

Some of the non-core elements may be lumped together according to the possible coincidence between the $\{\nu, \mu\}$ and $\{\nu', \mu'\}$ since the non-repetition rule do not prevent it. This procedure defines new partitions $\mathbf{D}$ that we will call *derived partitions*, denoted by $\mathbf{D} \succ \mathbf{D}_0$. In this particular case, there are seven possible lumpings of the four non-core elements without violating the non-repetition rule within the sets $(\tau, \nu, \mu)$ and $(\tau, \nu', \mu')$. Figure 10 shows the partition $\mathbf{D}$ derived from the above partition $\mathbf{D}_0$ when $\mu = \mu'$ but $\nu \neq \nu'$.

In general $\mathbf{D}$ is defined by lumping together a few non-core elements of $\mathbf{D}_0$ under the constraint of the non-repetition rule (Fig. 4). The single elements of $\mathbf{D}_0$ that correspond to $\theta$ are never lumped. Note that each of these pairings gives rise to the appearance of exactly four or six identical potential labels; higher moments do not appear. The number of such quartets and sextets is denoted by $\varrho_4(\mathbf{D})$ and $\varrho_6(\mathbf{D})$. Clearly $\varrho_4(\mathbf{D}) \le 2$ and $\varrho_6(\mathbf{D}) \le 1$.

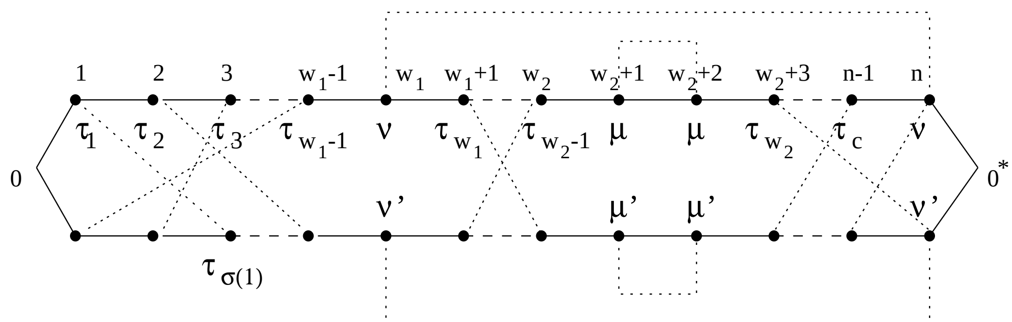


Figure 3: Symmetrized recollision with a gate. $\mathbf{D}_0$ consists of the paired vertices

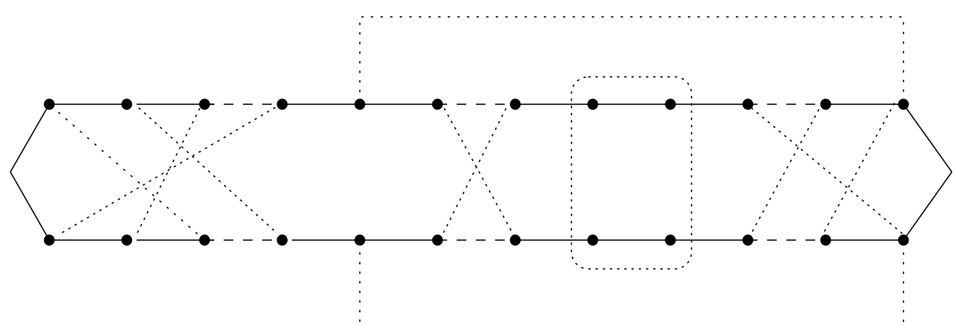

Figure 4: Partition $\mathbf{D}$ lumps some non-core elements of $\mathbf{D}_0$

Let $\mathbf{D}^* \subset \mathbf{D}$ denote the collection of non-single elements of $\mathbf{D}$. Note that for each element of $\mathbf{D}^*$ one selects a distinct potential label. The quantity (4.22) contains a summation over all such potential labels. We will use the connected graph formula (Lemma 6.1 from [10]) for the index set $\mathbf{D}^*$.

Let $\mathbf{A} \in \mathcal{A}(\mathbf{D}^*)$ be a partition of the set $\mathbf{D}^*$. We define $\mathbf{P}(\mathbf{A}, \mathbf{D}) \in \mathcal{P}_{n,n'}$ to be the partition of $I_n \cup \widetilde{I}_{n'}$ whose lumps are given by the equivalence relation that two elements of $I_n \cup \widetilde{I}_{n'}$ are $\mathbf{P}(\mathbf{A}, \mathbf{D})$-equivalent if their $\mathbf{D}$-lump(s) are $\mathbf{A}$-equivalent. The single lumps of $\mathbf{D}$ remain single in $\mathbf{P}$ (these are the $\theta$ indices).

We recall the definition of $V_{(*)}(\mathbf{P})$ and $V^\circ_{(*)}(\mathbf{P})$ from (4.14) and (4.17). Since we will compute the $L^2$-norm, the momentum shift at $0^*$ is chosen to be $\xi = 0$ (see (4.6)) and the $Q$ function in (4.12), representing the observable, will be $Q \equiv 1$. Furthermore, when defining Feynman graphs, we will always assume the following range of parameters (see (7.25) of [10]) unless stated otherwise:

$$\eta = \lambda^{2+\kappa}, \quad t = \lambda^{-2-\kappa} T, \quad T \in [0, T_0], \quad K = [\lambda^{-\delta}(\lambda^2 t)], \quad k \le K, \quad \zeta = \lambda^{-\kappa - 3\delta}, \quad g \le 8 \tag{4.23}$$

with a sufficiently small $\delta > 0$ that is independent of $\lambda$ but depends on $\kappa$. We recall that $\eta$ is the regularization of the propagator, $K$ is the upper treshold for the number of skeleton indices, $k$, in the expansion, $\zeta$ is the momentum cutoff (see (4.8)) and $g$ is the number of exceptional vertices where the standard $|\widehat{B}(w_{in} - w_{out})|$ potential decay is not present (this happens for the single lumps). All estimates will be uniform in $\xi$ and in $T \in [0, T_0]$.

For each fixed $(\#, c, \sigma, w, h, h')$, by using (1.5) and the connected graph formula, similarly to Proposition 7.2 of [10] we obtain

$$\lim_{L\to\infty} \sum_{\tau \in \Gamma_c^{nr}} \mathbf{E}' \Big\langle {\psi'}^{\,h,\#}_{(*)t,\tau,w}, {\psi'}^{\,h',\#}_{(*)t,\sigma(\tau),w} \Big\rangle = \sum_{\mathbf{D} \succ \mathbf{D}_0} \underline{m}^{\varrho(\mathbf{D})} \sum_{\mathbf{A} \in \mathcal{A}(\mathbf{D}^*)} c(\mathbf{A}) V^\circ_{(*)}(\mathbf{P}(\mathbf{A}, \mathbf{D})) \tag{4.24}$$

with $\underline{m}^{\varrho(\mathbf{D})} := m_4^{\varrho_4(\mathbf{D})} m_6^{\varrho_6(\mathbf{D})}$ (we recall $m_k = \mathbf{E}\, v_\gamma^k$ and (1.5)). The summary of the results in this section is

**Proposition 4.3** *Let $k \le K$, let $\# = rec, nest, triple, last$ and $r$ be one of the possible values allowed by (3.27) or $r = 1$ if $\# = nr$, $k < K$. Let $c$ be given by (4.19), and let $W = W_c^{(r),\#}$ be the set of location codes. Then*

$$\lim_{L\to\infty} \mathbf{E}' \| {\psi'}^{\,(r),\#}_{(*)t,k} \|^2 \le |W| \sum_{w \in W} \sum_{\sigma \in \mathfrak{S}_c} \sum_{h,h' \in \{g,\theta\}^r} \sum_{\mathbf{D} \succ \mathbf{D}_0} \underline{m}^{\varrho(\mathbf{D})} \sum_{\mathbf{A} \in \mathcal{A}(\mathbf{D}^*)} c(\mathbf{A}) V^\circ_{(*)}(\mathbf{P}(\mathbf{A}, \mathbf{D})) \, . \tag{4.25}$$

□

## 4.4 Splitting into high and low complexity regimes

Given $(\#, c, \sigma, w, h, h')$, we consider the partition $\mathbf{D}_0$ of $I_n \cup \widetilde{I}_{n'}$ as defined above and let $\mathbf{D} \succ \mathbf{D}_0$. Note that the collection $\mathbf{D}^*$ contains all core elements of $\mathbf{D}_0$, i.e. all pairs of core indices $\{s(j), \widetilde{s}(\sigma(j))\}_{j\in I_c}$. The restriction of a partition $\mathbf{A} \in \mathcal{A}(\mathbf{D}^*)$ onto these core elements can therefore be naturally identified with a partition of $I_c$ using the map $\{s(j), \widetilde{s}(\sigma(j))\}_{j\in I_c} \mapsto j \in I_c$. We denote this restricted partition by $\widehat{\mathbf{A}}$. In the sequel we shall therefore view $\widehat{\mathbf{A}} \in \mathcal{A}_c$, i.e. as a partition on $I_c$.

The restricted partition $\widehat{\mathbf{A}}$ together with $\sigma$ also generates a partition $\mathbf{P}(\widehat{\mathbf{A}}, \sigma)$ on the set $I_c \cup \widetilde{I}_c$. The lumps $P_\mu$ of $\mathbf{P}(\widehat{\mathbf{A}}, \sigma)$ are given by $\widehat{A}_\mu \cup \sigma(\widehat{A}_\mu)$, where $\widehat{A}_\mu$ are the lumps of $\widehat{\mathbf{A}}$. Notice that the $(s, \widetilde{s})$-image of the restriction of $\mathbf{P}(\mathbf{A}, \mathbf{D}) \in \mathcal{P}_{n,n'}$ onto the set of core indices $I_n^{core} \cup \widetilde{I}_{n'}^{core}$ is exactly $\mathbf{P}(\widehat{\mathbf{A}}, \sigma)$. Since the cardinality of non-core elements of $\mathbf{D}^*$ is at most 4, for any given $\mathbf{D}$ and $\widehat{\mathbf{A}}$ there can exist at most $(c+4)^4$ partitions, $\mathbf{A} \in \mathcal{A}(\mathbf{D}^*)$, whose restriction onto the core elements is $\widehat{\mathbf{A}}$.

We recall the definition of *joint degree* from [10]:

**Definition 4.4** *(i) Let $\mathbf{A} \in \mathcal{A}_k$ be a partition of $I_k = \{1, 2\ldots, k\}$. Set $a_\nu := |A_\nu|$, $\nu \in I(\mathbf{A})$, to be the size of the $\nu$-th lump. Let*

$$S(\mathbf{A}) := \bigcup_{\substack{\nu\in I(\mathbf{A})\\ a_\nu \geq 2}} A_\nu$$

*be the union of nontrivial lumps. The cardinality of this set, $s(\mathbf{A}) := |S(\mathbf{A})|$, is called the* **degree of the partition A**.

*(ii) Let $\mathbf{A} \in \mathcal{A}_k$ and $\sigma \in \mathfrak{S}_k$. The number*

$$q(\mathbf{A}, \sigma) := \max\Big\{\deg(\sigma), \frac{1}{2}s(\mathbf{A})\Big\} \tag{4.26}$$

*is called the* **joint degree** *of the pair $(\sigma, \mathbf{A})$ of the permutation $\sigma$ and partition* **A**.

The sum (4.25) will be split into two parts and estimated differently. In the regime of high combinatorial complexity, i.e., when the joint degree $q(\widehat{\mathbf{A}}, \sigma)$ of $\sigma$ and $\widehat{\mathbf{A}}$ is bigger than a threshold $q \geq 1$ (to be determined later), then we can use the method of Section 9 (especially Proposition 9.2) from [10] robustly. This will be explained in Section 4.5. For low combinatorial complexity we use the special structure given by the recollisions, nests, triple collisions or gates (Section 4.6). The threshold $q$ will be chosen differently for the estimates (4.1)–(4.2) and for (4.3).

The precise estimate is the following

$$\lim_{L\to\infty} \mathbf{E}' \|\psi'^{(r),\#}_{(*)t,k}\|^2 \leq (I) + (II) + O(\lambda^5) \tag{4.27}$$

with

$$(I) := |W|(c+4)^4 \sum_{w\in W} \sum_{h,h'} \sum_{\sigma\in\mathfrak{S}_c} \sum_{\mathbf{D}\succ\mathbf{D}_0} \underline{m}^{\varrho(\mathbf{D})} \sum_{\substack{\mathbf{A}'\in\mathcal{A}_c \\ q(\mathbf{A}',\sigma)\geq q}} \sup_{\mathbf{A}} \Big\{ |V_{(*)}(\mathbf{P}(\mathbf{A},\mathbf{D}))c(\mathbf{A})| \; : \; \widehat{\mathbf{A}} = \mathbf{A}' \Big\} \tag{4.28}$$

where the supremum is over all possible $\mathbf{A} \in \mathcal{A}(\mathbf{D}^*)$ whose restriction $\widehat{\mathbf{A}}$ is the given partition $\mathbf{A}'$; and

$$(II) := |W| \sum_{w\in W} \sum_{\sigma\in\mathfrak{S}_c} \Bigg| \sum_{h,h'\in\{g,\theta\}^r} \sum_{\mathbf{D}\succ\mathbf{D}_0} \underline{m}^{\varrho(\mathbf{D})} \sum_{\substack{\mathbf{A}\in\mathcal{A}(\mathbf{D}^*) \\ q(\widehat{\mathbf{A}},\sigma)<q}} V_{(*)}(\mathbf{P}(\mathbf{A},\mathbf{D}))c(\mathbf{A}) \Bigg| \; . \tag{4.29}$$

We recall that $\mathbf{D}_0$ is determined by $(\#, c, \sigma, w, h, h')$. The error term $O(\lambda^5)$ comes from replacing $V^{\circ}_{(*)}(\cdots)$ with $V_{(*)}(\cdots)$; see Lemma 7.1 of [10].

## 4.5 Case of high combinatorial complexity

Here we estimate the term (I) in (4.28). Clearly $\underline{m}^{\varrho(\mathbf{D})} \leq \langle m_4\rangle^2\langle m_6\rangle \leq C$. We estimate $V_{(*)}(\mathbf{P}(\mathbf{A},\mathbf{D}))$ by using (4.16). Then, by applying Operation I from Appendix C, we break up all the lumps $P_\mu$ in the partition $\mathbf{P}(\mathbf{A},\mathbf{D})$ that involve elements from non-core indices, $I_n^{nc} \cup \widetilde{I}_{n'}^{nc}$, in such a way that all non-core indices must form single lumps. Let $\mathbf{P}^*(\mathbf{A},\mathbf{D})$ denote this new partition. Note that the projection of $\mathbf{P}^*(\mathbf{A},\mathbf{D})$ onto the core indices is unchanged. The number of application of Operation I is at most 6. Using Lemma C.1, we can estimate $E_{(*)g}(\mathbf{P}(\mathbf{A},\mathbf{D}))$ in terms of $\sup_{\mathbf{u}} E_{(*)g}(\mathbf{P}^*(\mathbf{A},\mathbf{D}),\mathbf{u})$ with an additional factor of at most $\Lambda^6$, where

$$\Lambda := [CK\zeta]^d = O(\lambda^{-2d\kappa - O(\delta)}) \; .$$

Then we apply Operation II (Appendix C) to remove all single lumps with non-core indices and use (C.1) from Lemma C.2. After removing the non-core indices, the remaining vertex set can naturally be identified with $I_c \cup \widetilde{I}_c$ by using the $(s, \widetilde{s})$ maps, and the partition $\mathbf{P}^*(\mathbf{A},\mathbf{D})$ restricted to core indices $I_n^{core} \cup \widetilde{I}_{n'}^{core}$ is identified with $\mathbf{P}(\widehat{\mathbf{A}},\sigma)$. Lemma C.2 is applied at most 8 times, therefore we obtain that for any $\sigma$, $\mathbf{D}$ and $\mathbf{A}'$ in the sum (4.28):

$$|V_{(*)}(\mathbf{P}(\mathbf{A},\mathbf{D}))| \leq C\Lambda^6(\lambda\eta^{-1})^8 \sup_{\mathbf{u},g\leq 8} E_{(*)g}(\mathbf{A}',\sigma,\mathbf{u}) \; . \tag{4.30}$$

The application of Operation II is schematically shown on Fig. 5.

The summations over $h, h' \in \{g,\theta\}^r$ and $\mathbf{D} \succ \mathbf{D}_0$ in (4.28) contribute with at most a constant factor since $r \leq 2$ and the number of different $\mathbf{D}$'s is at most 7. The cardinality

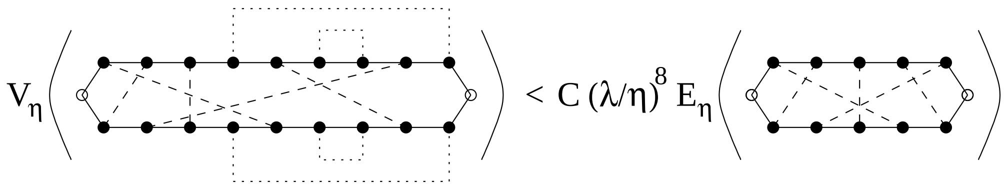


Figure 5: Estimate after removing all non-core indices

of $W$ can be bounded by $(c+1)^2$ and $c \le K \le C\lambda^{-\kappa-\delta}$. Therefore we obtain

$$(I) \le C\lambda^{-8-\kappa(16+2d)-O(\delta)} \sum_{\sigma\in\mathfrak{S}_c} \sum_{\substack{\mathbf{A}'\in\mathcal{A}_c \\ q(\mathbf{A}',\sigma)\ge q}} \sup_{\mathbf{u},g\le 8} E_{(*)g}(\mathbf{A}',\sigma,\mathbf{u})|c(\mathbf{A}')| \;.$$

Using Proposition 9.2 from [10], we have

$$(I) \le C\lambda^{q[\frac{1}{3}-(\frac{17}{3}d+\frac{13}{2})\kappa-O(\delta)]-8-(16+2d)\kappa} \;.$$

We immediately see, that the contribution of (I) to the error term in (4.3) in Theorem 4.1 satisfies the announced bound with a sufficiently small $\delta$ if

$$\kappa < \frac{2q-48}{(34d+39)q+112+12d} \;. \tag{4.31}$$

The bounds (4.1)–(4.2) are satisfied if

$$\kappa < \frac{2q-72}{(34d+39)q+112+12d} \;. \tag{4.32}$$

## 4.6 Case of small combinatorial complexity

Here we control the term (II) in (4.29). First we estimate the combinatorics.

**Lemma 4.5** *For any $q\in\mathbf{N}$, $c\le K$ and structure type $\#$, we have*

$$\sup_{w,h,h'} \sum_{\sigma\in\mathfrak{S}_c} \sup_{\mathbf{D}\succ\mathbf{D}_0} \sum_{\substack{\mathbf{A}\in\mathcal{A}(\mathbf{D}^*) \\ q(\widehat{\mathbf{A}},\sigma)<q}} |c(\mathbf{A})| \le (CqK)^{3q+3} \;,$$

*where we recall that $\mathbf{D}_0$ depends on $(\#, c, \sigma, w, h, h')$.*

*Proof.* The bound

$$\#\{\sigma\in\mathfrak{S}_k \;:\; \ell(\sigma)=\ell\} \le (Ck)^{k-\ell+1} \tag{4.33}$$

(see (8.14) from [10]) shows that the number of permutations $\sigma \in \mathfrak{S}_c$ with $\deg(\sigma) < q$ is bounded by $(CK)^q$ using $c \le K$. The number of $\mathbf{A}$'s whose restriction yields the same $\widehat{\mathbf{A}}$ is at most $(c+3)^4 \le (CK)^4$. The number of $\widehat{\mathbf{A}} \in \mathcal{A}_c$ with $s(\widehat{\mathbf{A}}) < 2q$ is bounded by $c^{2q-1} \le (CK)^{2q-1}$. Finally, $|c(\mathbf{A})| \le \prod_j^* a_j^{a_j - 2} \le (2q)^{2q-2}$ $\square$.

The individual terms in (4.29) are estimated in the following Proposition whose proof will be given in Section 5.

**Proposition 4.6** *We assume (4.23) and $\kappa < \frac{2}{34d+39}$. Let $\sigma \in \mathfrak{S}_c$, $w \in W_c^{(r),\#}$, $h, h' \in \{g, \theta\}^r$, where $\#$ and $r$ vary in the different cases and let $\mathbf{D} \succ \mathbf{D}_0(\#, c, \sigma, w, h, h')$.*

*1) Let $\mathbf{A} \in \mathcal{A}(\mathbf{D}^*)$ such that $q(\widehat{\mathbf{A}}, \sigma) < q$, where $q$ is a fixed number. Then the following individual estimates hold.*

*(1a) [Many collisions] Let $\# = nr$, $r = 0, 1$ and $c = K$, then*

$$|V_*(\mathbf{P}(\mathbf{A}, \mathbf{D}))| \le C^q \lambda^{\frac{\delta}{2}K} . \tag{4.34}$$

*(1b) [Recollision]. Let $\# = rec$, $r = 0, 1$, then*

$$|V_*(\mathbf{P}(\mathbf{A}, \mathbf{D}))| \le C^q \lambda^{6-4d\kappa(q+3)} . \tag{4.35}$$

*(1c) [Triple collision] Let $\# = triple$, $r = 1$, then*

$$|V_*(\mathbf{P}(\mathbf{A}, \mathbf{D}))| \le C^q \lambda^{6-4d\kappa(q+3)} . \tag{4.36}$$

*2) Now let $\mathbf{A}' \in \mathcal{A}_c$ be given. Then the following estimates hold:*

*(2a) [Non-repetition with a gate] Let $\# = nr$, $r = 1$, then*

$$\sup_{\sigma, w} \Big| \sum_{h,h' \in \{g,\theta\}^r} \sum_{\mathbf{D} \succ \mathbf{D}_0} \sum_{\substack{\mathbf{A} \in \mathcal{A}(\mathbf{D}^*) \\ \widehat{\mathbf{A}} = \mathbf{A}'}} V(\mathbf{P}(\mathbf{A}, \mathbf{D})) c(\mathbf{A}) \Big| \le C \lambda^{\frac{1}{3} - (\frac{17}{3}d + 8)\kappa - O(\delta)} . \tag{4.37}$$

*(2b) [Last] Let $\# = last$, $r = 2$, then*

$$\sup_{\sigma, w} \Big| \sum_{h,h' \in \{g,\theta\}^r} \sum_{\mathbf{D} \succ \mathbf{D}_0} \sum_{\substack{\mathbf{A} \in \mathcal{A}(\mathbf{D}^*) \\ \widehat{\mathbf{A}} = \mathbf{A}'}} V_*(\mathbf{P}(\mathbf{A}, \mathbf{D})) c(\mathbf{A}) \Big| \le C \lambda^{6-(14d+6)\kappa - O(\delta)} . \tag{4.38}$$

*(2c) [Nest] Let $\# = nest$, $r = 1$, then*

$$\sup_{\sigma, w} \Big| \sum_{h,h' \in \{g,\theta\}} \sum_{\mathbf{D} \succ \mathbf{D}_0} \sum_{\substack{\mathbf{A} \in \mathcal{A}(\mathbf{D}^*) \\ \widehat{\mathbf{A}} = \mathbf{A}'}} V_*(\mathbf{P}(\mathbf{A}, \mathbf{D})) c(\mathbf{A}) \Big| \le C \lambda^{6-(10d+8)\kappa - O(\delta)} . \tag{4.39}$$

Combining Lemma 4.5 with these estimates, and using $|W| \leq K^2$, we see that

$$(II) \leq (CqK)^{3q+7} \lambda^{\frac{1}{3} - (\frac{17}{3}d+8)\kappa - O(\delta)}$$

for the case $\# = nr$, $r = 1$ (case (2a) above), and

$$(II) \leq (CqK)^{3q+7} \lambda^{6-4d\kappa(q+3)-O(\delta)}$$

for all other cases (with $q \geq 2$). So the contributions of the error terms from (II) to $\mathbf{E}' \|\psi'^{(r),\#}_{(*)t,k}\|^2$ (see (4.27)) satisfy the bound (4.3) if

$$\kappa < \frac{1}{9q + 17d + 51}\ , \tag{4.40}$$

and they satisfy (4.1)-(4.2) if

$$\kappa < \frac{2}{(4d+3)q + (12d+9)} \tag{4.41}$$

and $\delta$ is sufficiently small. Combining this with (4.31)–(4.32) and optimizing, we obtain that there exists $\kappa_0(d) > 0$ such that for any $\kappa < \kappa_0$, the systems of inequalities (4.31)–(4.40) and (4.32)–(4.41) have solutions for $q$. For $d = 3$, the optimal $\kappa_0(d)$ is a bit above $\frac{1}{500}$. This finishes the proof of Theorem 4.1. $\quad\square$.

# 5 Proof of Proposition 4.6

In each case except (1a), the corresponding Feynman graph has a specific subgraph of a few vertices (recollision, nest etc.) that renders the value small. We shall prove that this subgraph gains at least a factor $\lambda^{4+2\kappa+2\delta}$ required in Theorem 4.1. Then we remove all repetition patterns from the graph, we use the robust bounds

$$\sup_{\sigma \in \mathfrak{S}_k} \sup_{\mathbf{u}} E(\sigma, \mathbf{u}) \leq C|\log \lambda|^2 \tag{5.1}$$

$$\sup_{\sigma \in \mathfrak{S}_k} \sup_{\mathbf{u}} E_*(\sigma, \mathbf{u}) \leq C\lambda^2 |\log \lambda|^2 \tag{5.2}$$

from Lemma 10.2. of [10] to conclude the estimate.

## 5.1 Many collisions

The estimate in case (1a) will come from the fact that any graph can be robustly estimated by the ladder graph and the value of the ladder of length $L$ always carries a factor $1/L!$. This effect is the best seen in the time integral form. We first change $V_*(\mathbf{P})$ back to $V_*^\circ(\mathbf{P})$ with an error smaller than $O(\lambda^{10K-O(1)})$ by Lemma 7.1 from [10]. We then apply the $K$-identity (formula (6.2) in [10]) to the definition of $V_*^\circ(\mathbf{P})$ given in (4.17) to obtain

$$V_*^\circ(\mathbf{P}) = \lambda^{n+n'+g} \iint \mathrm{d}\mathbf{p}\mathrm{d}\tilde{\mathbf{p}} \, \overline{K(t,\mathbf{p},n)} K(t,\tilde{\mathbf{p}},n') \Delta(\mathbf{P},\mathbf{w},\mathbf{u}\equiv 0)\mathcal{M}(\mathbf{w})$$

with $\mathbf{P} = \mathbf{P}(\mathbf{A},\mathbf{D})$ and

$$K(t,\mathbf{p},n) := (-i)^{n-1} \int_0^t [\mathrm{d}s_j]_1^n \prod_{j=1}^n e^{-is_j\omega(p_j)} \ .$$

We recall that $g = g(\mathbf{P}(\mathbf{A},\mathbf{D}))$ denotes the number of single lumps, or, equivalently, the number of $\theta$ labels in $h$ and $h'$. Note that we use the labelling $\mathbf{w}$ and $\mathbf{p},\tilde{\mathbf{p}}$ in parallel, keeping in mind the relabelling convention from (4.18) (see also Section 7.2 in [10] for more details).

We use a Schwarz inequality:

$$|V_*^\circ(\mathbf{P})| \le \lambda^{n+n'+g} \iint \mathrm{d}\mathbf{p}\mathrm{d}\tilde{\mathbf{p}} \, \Big[|K(t,\mathbf{p},n)|^2 + |K(t,\tilde{\mathbf{p}},n')|^2\Big] \Delta(\mathbf{P},\mathbf{w},\mathbf{u}\equiv 0)|\mathcal{M}(\mathbf{w})| \ .$$

By using Operation I, we can break up $\mathbf{A}$ into single lumps. Since $s(\widehat{\mathbf{A}}) \le 2q$, we have $s(\mathbf{A}) \le 2q+4$, thus Operation I will be applied at most $2q+3$ times.

If $r=0$, i.e. the original graph was a non-repetition graph, and thus $n = n' = k$, then the trivial partition $\mathbf{A}_0$ corresponds to a partition $\mathbf{P}$ with a complete pairing. Thus both $\mathbf{p}$ and $\tilde{\mathbf{p}}$ momenta can be used as independent variables and

$$|V_*^\circ(\mathbf{P})| \le \Lambda^{2q+3}(C\lambda)^{2k+g} \int \mathrm{d}\mathbf{p} \, |K(t,\mathbf{p},k)|^2 |\widehat{\psi}_0(p_1)|^2 \prod_{j=1}^k |\widehat{B}(p_j - p_{j+1})|^2 \ .$$

The estimate (4.34) is then completed by the bound (5.3) from the following Lemma with any $1/2 < a < 1$. The proof will be given below.

**Lemma 5.1** *For any $0 \le a < 1$, $t = T\lambda^{-2-\kappa}$, there exists a constant $C_a$ such that*

$$I(k) := \int \mathrm{d}\mathbf{p} |K(t,\mathbf{p},k)|^2 |\widehat{\psi}_0(p_1)|^2 \prod_{j=1}^k |\widehat{B}(p_j - p_{j+1})|^2 \le \frac{(C_a T\lambda^{-2-\kappa a})^{k-1}}{[(k-1)!]^a} |\log\lambda|^2 \ .$$

*In particular,*

$$I(k) \le (C_a \lambda^{-2+\delta a})^{k-1} \tag{5.3}$$

*if $k \ge T\lambda^{-\kappa-\delta}$ and $\lambda \ll 1$.*

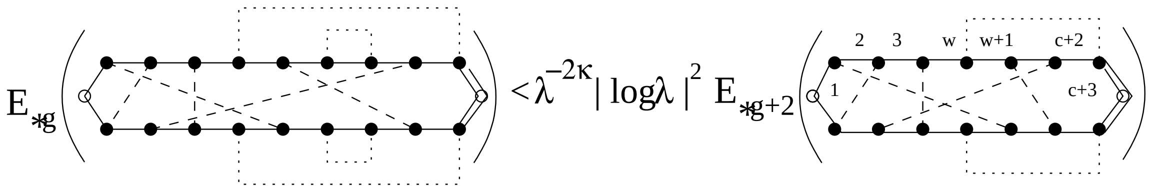


Figure 6: Removal of gates from a recollision

*Proof.* This lemma is essentially the same as Lemma 3.1 in [7]. The only differences are that here we estimate the truncated value, so $K$ has one less time integration and the individual integrals are performed by using Lemma 2.1. The details are left to the reader. □

Finally, if $r = 1$ and $w \in W_c^{(1),nr}$ is the location of the gate/$\theta$-index among the core indices, then $p_w = p_{w+1}$ or $p_w = p_{w+2}$ (depending whether we have a $\theta$ or a gate) is forced by $\Delta$ and similarly for $\tilde{p}_w$. In this case, the estimates in Lemma 5.1 are worse by a factor of $t$. This factor can be absorbed into the main term $\lambda^{\delta a K}$. This completes the proof of (4.34). □

## 5.2 Recollision and triple collision

For the proof of (4.35), we break up the partition $\mathbf{A}$ into the trivial partition $\mathbf{A}_0$ using Operation I. Since $s(\widehat{\mathbf{A}}) \leq 2q$, we have $s(\mathbf{A}) \leq 2q + 4$, thus Operation I will be applied at most $2q + 3$ times. Clearly

$$|V_*(\mathbf{P}(\mathbf{A}, \mathbf{D}))| \leq \Lambda^{2q+3} \lambda^g \sup_{\mathbf{u}} E_*(\mathbf{P}(\mathbf{A}_0, \mathbf{D}_0), \mathbf{u}) \tag{5.4}$$

by using (4.16) and Lemma 9.5 from [10]. The single lumps are removed by Operation II from $E_*(\mathbf{P}(\mathbf{A}_0, \mathbf{D}_0), \mathbf{u})$, at the price $\lambda/\eta$; the total contribution of one $\theta$-removal is $(\lambda^2/\eta) \sim \lambda^{-\kappa}$. The possible gates are eliminated by Operation IV at the expense of $\lambda^2 \eta^{-1} |\log \eta| \sim \lambda^{-\kappa} |\log \lambda|$ each. Since $r \leq 1$, we lose at most a factor $\lambda^{-2\kappa} |\log \eta|^2$ in this way (see Fig. 6; the double line denotes truncated propagators).

The vertex set of the remaining graph is naturally identified with $I_{c+2} \cup \widetilde{I}_{c+2} \cup \{0, 0^*\}$, the permutation $\sigma$ provides a pairing between the elements $I_{c+2} \setminus \{w_1, c+2\}$ and $\widetilde{I}_{c+2} \setminus \{\widetilde{w_1}, \widetilde{c+2}\}$, furthermore $\{w_1, c+2\}$ and $\{\widetilde{w_1}, \widetilde{c+2}\}$ each form a lump (see picture). We denote this partition by $\mathbf{P}^*$ and by using Proposition 5.2 below, we will obtain (4.35). □

Later we need to estimate asymmtric recollision graphs as well, so we formulate the following proposition in a more general setup:

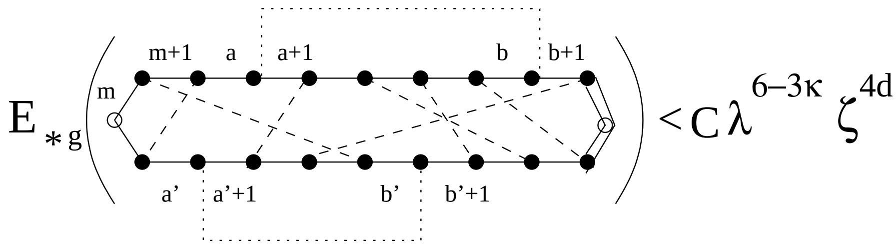


Figure 7: Estimate of a two-sided recollision graph

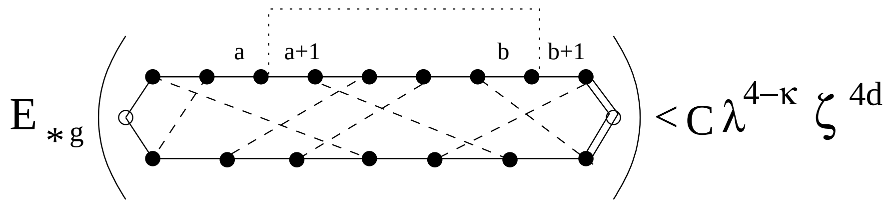


Figure 8: Estimate of a one-sided recollision graph

**Proposition 5.2** *Consider the Feynman graph on the vertex set $\mathcal{V}_k$, $k \geq 3$, choose numbers $a, b, a', b' \in I_k$ such that $b - a \geq 2$, $b' - a' \geq 2$. Let $\sigma$ be a bijection between $I_k \setminus \{a, b\}$ and $\widetilde{I}_k \setminus \{a', b'\}$. Let $\mathbf{P}^*$ be the partition on the set $I_k \cup \widetilde{I}_k$ consisting of the lumps $\{j, \sigma(j)\}$, $j \in I_k \setminus \{a, b\}$ and $\{a, b\}$, $\{a', b'\}$ (Fig. 7). Then*

$$\sup_{\mathbf{u}, g \leq 8} E_{*g}(\mathbf{P}^*, \mathbf{u}) \leq C\lambda^{6-3\kappa}\zeta^{4d} \, . \tag{5.5}$$

We also need a "one-sided" version of this estimate (Fig. 8).

**Proposition 5.3** *Consider the Feynman graph on the vertex set $\mathcal{V}_{k,k-2}$, $k \geq 3$, choose numbers $a, b \in I_k$ such that $b - a \geq 2$. Let $\sigma$ be a bijection between $I_k \setminus \{a, b\}$ and $\widetilde{I}_{k-2}$. Let $\mathbf{P}^*$ be the partition on the set $I_k \cup \widetilde{I}_{k-2}$ consisting of the lumps $\{j, \sigma(j)\}$, $j \in I_k \setminus \{a, b\}$ and $\{a, b\}$. Then*

$$\sup_{\mathbf{u}, g \leq 8} E_g(\mathbf{P}^*, \mathbf{u}) \leq C\lambda^{2-\kappa}\zeta^{4d} \, , \tag{5.6}$$

*and for the truncated version*

$$\sup_{\mathbf{u}, g \leq 8} E_{*g}(\mathbf{P}^*, \mathbf{u}) \leq C\lambda^{4-\kappa}\zeta^{4d} \, . \tag{5.7}$$

*Proof of Propostion 5.2.* We use $\mathbf{p} = (p_1, \dots, p_{k+1})$ and their tilde-counterparts to denote the momenta to express

$$E_{*g}(\mathbf{P}^*, \mathbf{u}) = \sup_{\mathcal{G}\,:\,|\mathcal{G}|\le g} \int_{-Y}^{Y} \mathrm{d}\alpha \mathrm{d}\beta \, \Xi_{\mathcal{G}}(\alpha,\beta)$$

with

$$\Xi_{\mathcal{G}}(\alpha,\beta) := \lambda^{2k} \int \mathrm{d}\mu(\mathbf{p})\mathrm{d}\mu(\tilde{\mathbf{p}}) \prod_{j=1}^{k} \frac{1}{|\alpha - \overline{\omega}(p_j) - i\eta|} \frac{1}{|\beta - \omega(\tilde{p}_j) + i\eta|}$$

$$\times \delta(p_{k+1} - \tilde{p}_{k+1}) \delta\Big(p_{a+1} - p_a + (p_{b+1} - p_b) - u_a\Big) \delta\Big( - \tilde{p}_{a'+1} + \tilde{p}_{a'} - (\tilde{p}_{b'+1} - \tilde{p}_{b'}) - \tilde{u}_{a'}\Big)$$

$$\times \prod_{\substack{j=1 \\ j\neq a,b}}^{k} \delta\Big(p_{j+1} - p_j - (\tilde{p}_{\sigma(j)+1} - \tilde{p}_{\sigma(j)}) - u_j\Big) \, \mathcal{N}_{\mathcal{G}}(\mathbf{w}) \,,$$

where the $\mathbf{u}$-momenta are labelled as $\mathbf{u} = (u_1, \dots, u_{b-1}, u_{b+1}, u_k, \tilde{u}_{a'})$ and we used the identification from (4.18) between the $\mathbf{w}$ and $\mathbf{p}, \tilde{\mathbf{p}}$ notations.

Without recollision, the momenta $\mathbf{p}$ formed a spanning set of all momenta, and similarly for $\tilde{\mathbf{p}}$. Since now there is a delta function among the $\mathbf{p}$ momenta, we need to exchange one tilde-momentum (out of $\tilde{p}_{a'}, \tilde{p}_{a'+1}, \tilde{p}_{b'}, \tilde{p}_{b'+1}$) with a non-tilde momentum (out of $p_a, p_{a+1}, p_b, p_{b+1}$). We will call them *exchange momenta*.

For the moment, we choose $\tilde{p}_{b'}$ and $p_b$ to be the exchange momenta and we partition the set of all $\mathbf{p}, \tilde{\mathbf{p}}$ momenta into two subsets of size $k+1$ each:

$$A := \{p_1, p_2, \dots, p_{b-1}, p_{b+1}, \dots p_{k+1}, \tilde{p}_{b'}\}$$

$$B := \{\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{b'-1}, \tilde{p}_{b'+1}, \dots \tilde{p}_{k+1}, p_b\} \,.$$

It is straightforward to check that all $A$-momenta can be uniquely expressed in terms of linear combinations of the $B$-momenta (plus the $\mathbf{u}$-momenta) and conversely. In particular

$$p_{b-1} = p_b - (\tilde{p}_{\sigma(b-1)+1} - \tilde{p}_{\sigma(b-1)}) - u_b$$

$$\tilde{p}_{b'-1} = \tilde{p}_{b'} - (p_{m+1} - p_m) + u_m \quad \text{with} \qquad m := \sigma^{-1}(b'-1) \,.$$

The letters on the pictures indicate the indices of the correspoding $p$ or $\tilde{p}$ momenta.

We perform a Schwarz estimate to separate $A$ and $B$-momenta at the expense of squaring the propagators, but we keep the denominators with $p_1, \tilde{p}_1, p_{b-1}, \tilde{p}_{b'-1} p_b, \tilde{p}_{b'}$ common and only on the first power:

$$\prod_{j=1}^{k} \frac{1}{|\alpha - \overline{\omega}(p_j) - i\eta|} \frac{1}{|\beta - \omega(\tilde{p}_j) + i\eta|} \le \frac{1}{2}\big[(a) + (b)\big] \tag{5.8}$$

$$(a) := \prod_{j=1,b-1,b} \frac{1}{|\alpha - \overline{\omega}(p_j) - i\eta|} \frac{1}{|\beta - \omega(\tilde{p}_{j'}) + i\eta|} \prod_{\substack{j=2 \\ j \neq b-1,b}}^{k} \frac{1}{|\alpha - \overline{\omega}(p_j) - i\eta|^2}$$

$$(b) := \prod_{j=1,b-1,b} \frac{1}{|\alpha - \overline{\omega}(p_j) - i\eta|} \frac{1}{|\beta - \omega(\tilde{p}_{j'}) + i\eta|} \prod_{\substack{j=2 \\ j \neq b'-1,b'}}^{k} \frac{1}{|\beta - \omega(\tilde{p}_j) + i\eta|^2} .$$

(with a little abuse of notations we used $j'$ for 1, $b'-1$ and $b'$ when $j = 1, b-1$ and $b$, respectively).

Since common factors can be explicitly expressed both in terms of $A$ and $B$-momenta, we can compute the integral of (a) by first integrating all $B$-momenta that removes all delta functions, then estimating the $A$-momentum integrals. Similar procedure works for (b). The result is (with $m := \sigma^{-1}(b'-1)$)

$$\mathbf{E}_{*g}(\mathbf{P}^*, \mathbf{u}) \leq \lambda^6 \sup_{|\mathcal{G}| \leq g} \iint_{-Y}^{Y} \mathrm{d}\alpha \mathrm{d}\beta \int \mathrm{d}\mu(\tilde{p}_{b'}) \Big( \prod_{\substack{j=1 \\ j \neq b}}^{k+1} \mathrm{d}\mu(p_j) \Big) \mathcal{N}_{\mathcal{G}}(\mathbf{w}) \frac{1}{|\alpha - \overline{\omega}(p_1) - i\eta|} \tag{5.9}$$
$$\times \frac{1}{|\beta - \omega(p_1) + i\eta|} \frac{1}{|\alpha - \overline{\omega}(p_{b-1}) - i\eta|} \frac{1}{|\beta - \omega(\tilde{p}_{b'} - (p_{m+1} - p_m) + u_m) + i\eta|}$$
$$\times \frac{1}{|\alpha - \overline{\omega}(p_{b+1} - p_a + p_{a+1} - u_a) - i\eta|} \frac{1}{|\beta - \omega(\tilde{p}_{b'}) + i\eta|} \prod_{\substack{j=2 \\ j \neq b-1,b}}^{k} \frac{\lambda^2}{|\alpha - \overline{\omega}(p_j) - i\eta|^2} .$$

We recall the key technical bound Lemma 10.8 from [10] to estimate integrals with shifted denominators. We also recall the notation $|\!|\!| q |\!|\!| := \eta + \min\{|q|, 1\}$. We start with estimating

$$\frac{\lambda^2}{|\alpha - \overline{\omega}(p_{b+1}) - i\eta|^2} \leq \frac{\lambda^2 \eta^{-1}}{|\alpha - \overline{\omega}(p_{b+1}) - i\eta|}$$

and integrating out $p_{b+1}$ by using (10.25) from [10]. We collect a point singularity $|\!|\!| p_a - p_{a+1} + u_a |\!|\!|^{-1}$ and a factor $C\lambda^2\eta^{-1}\zeta^{d-3}|\log\eta|^2$. This argument works if $b < k$; the $b = k$ case is even easier since the denominator with $p_{k+1}$ is not present. Then we perform the $\tilde{p}_{b'}$ integration again by (10.25) from [10], and we collect a new point singularity $|\!|\!| p_{m+1} - p_m + u_m |\!|\!|^{-1}$ and a factor $C\zeta^{d-3}|\log\lambda|^2$. Note that these two point singularities are not identical, because $m$, as an inverse image of $\sigma$, is not equal to $a$.

Next we integrate out $p_{b-1}$ yielding $C|\log\lambda|$ from (2.6). If $p_{b-1}$ appears in one of the point singularities, then we use the bound

$$\sup_{\alpha, r} \int \frac{\mathrm{d}\mu(p)}{|\alpha - \omega(p) + i\eta|} \frac{1}{|\!|\!| p - r |\!|\!|} \leq C\zeta^{d-2}|\log\eta| \tag{5.10}$$

(see (A.3) and (A.7) from [10]) to collect $C\zeta^{d-2}|\log\lambda|$ and the point singularity disappears. If $p_{b-1}$ appears in both point singularities, then we separate them by the telescopic estimate (A.1) of [10] before applying (5.10).

Now we integrate out all $p_j$'s with $j \neq 1, b-1, b, b+1$ in decreasing order with the successive integration scheme (10.7)–(10.9) from Section 10.1.2 of [10]. The factor $\mathcal{N}_{\mathcal{G}}(\mathbf{w})$ provides the necessary $|\widehat{B}(p_j - p_{j-1})|^2$ terms with at most eight exceptions, namely when $j-1 \in \mathcal{G}$ or $\sigma(j-1) \in \mathcal{G}$ (recall $|\mathcal{G}| \leq 8$). At each exceptional index $|\widehat{B}(p_j - p_{j-1})|^2$ is replaced with $\langle p_j - p_{j-1}\rangle^{-2d}$ and we use (2.7) instead of (10.8) of [10]. Thus the successive production of the factors $(1+C\lambda^{1-12\kappa})$ breaks at these indices and we obtain a uniform constant $C$ instead. The successive scheme also breaks at the indices $j = b-1, b, b+1$ that have already been integrated out. Furthermore, it also may break at $j = m+1, a+1$, i.e., at the indices where the point singularities are first affected (unless $b-1 \in \{m+1, a+1\}$ and the point singularity has already been integrated out). At each of these indices we use (5.10) and collect $C\lambda^2\eta^{-1}\zeta^{d-2}|\log\eta|^2$ instead of the constant factor from (10.7)–(10.9) of [10].

Since there are at most 13 exceptional indices, so we collect at most

$$C^{13}[\lambda^2\eta^{-1}\zeta^{d-2}|\log\eta|^2]^2(1+C\lambda^{1-12\kappa})^K.$$

The other factors of $\mathcal{N}_{\mathcal{G}}$, that are not explicitly used in the successive integration, are estimated by supremum norm, except $|\widehat{\psi}_0(p_1)|^2$. Finally the $\mathrm{d}\alpha$, $\mathrm{d}\beta$ integrals contribute with an additional $C|\log\lambda|^2$. The last $p_1$-integral is finite by the factor $|\widehat{\psi}_0(p_1)|^2$. Collecting these estimates and recalling that (5.10) has been used twice, the result is (5.5). $\square$

*Proof of Proposition 5.3.* This proof is very similar to the previous one but the estimate is weaker since (10.25) from [10] can be used only once. We just indicate that the set of $A$ and $B$ momenta are as follows:

$$A := \{p_1, p_2, \ldots, p_{b-1}, p_{b+1}, \ldots, p_{k+1}\}, \qquad B := \{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{k-1}, p_b\},$$

and we leave the details to the reader. $\square$

The case of the triple collision, (4.36), can be easily reduced to the case of a recollision (Fig. 9). We first use the analogue of the estimate (5.4). Clearly $g = 0$ in case of a triple collision. Then we remove half of each of the two gates (Lemma C.3) and we collect a factor $\lambda^2|\log\lambda|^2$. The resulting Feynman graph has either a recollision or a gate at the end. In the first case we apply Proposition 5.2. In the second case, we remove half of each gates by a second application of Lemma C.3, then the estimate (C.2) from Lemma C.2 together with (5.1) can be applied. $\square$

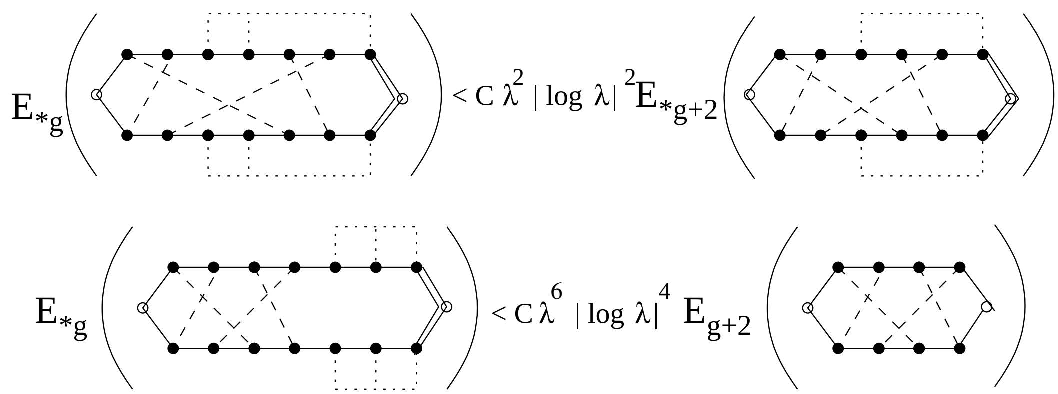


Figure 9: Estimate of triple collisions for $w < c+1$ and $w = c+1$

## 5.3 Cancellation with a gate

The key mechanism behind the estimates (4.37)–(4.39) is the cancellation between a gate and a $\theta$-label, We first present estimates on general graphs.

### 5.3.1 Cancellation between a gate and $\theta$

Fix $n, n'$ integers and consider a partition $\mathbf{P} \in \mathcal{P}_{n,n'}$ with no single lump on the set $I_n \cup \widetilde{I}_{n'}$ within the vertex set $\mathcal{V} = \mathcal{V}_{n,n'}$. Let $1 \le m \le n+1$ an integer. We define two new cyclically ordered sets:

$$\mathcal{V}' := \{0, 1, 2, \ldots, m-1, \clubsuit, m, \ldots, n, 0^*, \tilde{n}', \widetilde{n'-1}, \ldots, \tilde{1}\} \tag{5.11}$$

$$\mathcal{V}'' := \{0, 1, 2, \ldots, m-1, \diamondsuit, \heartsuit, m, \ldots, n, 0^*, \tilde{n}', \widetilde{n'-1}, \ldots, \tilde{1}\}$$

with additional elements $\clubsuit, \diamondsuit, \heartsuit$. For the result of this section it would make no difference if the extra elements were inserted into the sequence of tilde-variables.

These sets can be naturally identified with $\mathcal{V}_{n+1,n'}$ and $\mathcal{V}_{n+2,n'}$ and we will use this identification with the obvious choice of the relabelling map. We define two partitions on these sets, $\mathbf{P}' \in \mathcal{P}_{n+1,n'}$ and $\mathbf{P}'' \in \mathcal{P}_{n+2,n'}$, simply by adding the single lump $\{\clubsuit\}$ to $\mathbf{P}$ in the first case and the double lump $\{\diamondsuit, \heartsuit\}$ in the second case. This will correspond to adding a $\vartheta$ label or a gate whose potential labels have been paired to the original partition $\mathbf{P}$, respectively. The following lemma shows that the $V$-value of these two partitions cancel each other up to the lowest order (Fig. 10).

**Lemma 5.4** *With the notations above and assuming $\lambda^3 \ll \eta \ll \lambda^2$, we have*

$$\Big| V_{(*)}(\mathbf{P}') + V_{(*)}(\mathbf{P}'') \Big| \le C\lambda^2 \eta^{-1/2} E_{(*)g=0}(\mathbf{P}) \,. \tag{5.12}$$

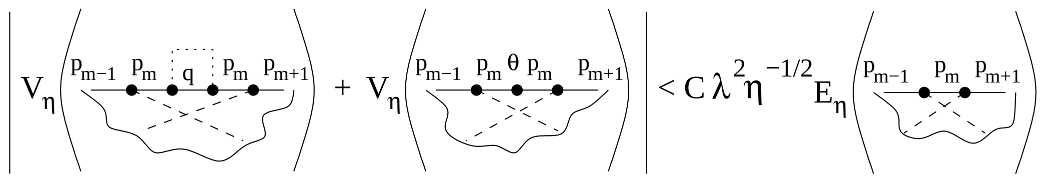


Figure 10: Cancellation of a gate and $\theta$ (partition is the same elsewhere)

*Proof of Lemma 5.4.* Introduce the notations $\mathbf{p} := (p_1, \ldots, p_{n+1})$, $\tilde{\mathbf{p}} := (\tilde{p}_1, \ldots, \tilde{p}_{n'+1})$,

$$\int \mathrm{d}\mathbf{p} := \iint \mathrm{d}p_1 \mathrm{d}p_2 \ldots \mathrm{d}p_{n+1}$$

and similarly for $\int \mathrm{d}\tilde{\mathbf{p}}$. Then we have

$$V_{(*)}(\mathbf{P}') + V_{(*)}(\mathbf{P}'') = \lambda^{n+n'+g(\mathbf{P})} \frac{e^{2t\eta}}{(2\pi)^2} \iint_{-Y}^{Y} \mathrm{d}\alpha \mathrm{d}\beta \; e^{i(\alpha-\beta)t} \int \mathrm{d}\mathbf{p} \; \mathrm{d}\tilde{\mathbf{p}} \; \Delta(\mathbf{P}, \mathbf{w}, \mathbf{u} \equiv 0) \mathcal{M}(\mathbf{w})$$

$$\times \Omega(\alpha, p_m) \prod_{j=1}^{n+1,\,(n)} \frac{1}{\alpha - \overline{\omega}(p_j) - i\eta} \prod_{j=1}^{n'+1,\,(n')} \frac{1}{\beta - \omega(\tilde{p}_j) + i\eta} \,, \tag{5.13}$$

with

$$\Omega(\alpha, p_m) := \left[ \int \frac{|\widehat{B}(q - p_m)|^2 \mathrm{d}q}{\alpha - \overline{\omega}(q) - i\eta} - \overline{\theta}(p_m) \right] \frac{\lambda^2}{\alpha - \overline{\omega}(p_m) - i\eta} \,. \tag{5.14}$$

The expression $n+1, (n)$ on the product sign indicates that for the truncated values (*) the last fraction is not present, i.e. $j$ runs up to $n$. Notice that the sum of the contributions of a gate and a $\vartheta$ inserted between $m$ and $m-1$ yields an additional factor $\Omega(\alpha, p_m)$ in the $V$-value of the $\mathbf{P}$ partition. Using (B.1) and (B.9) with $\varepsilon = \eta$, $\varepsilon' \to 0+0$, we have

$$\Big| \int \frac{|\widehat{B}(q - p_m)|^2 \mathrm{d}q}{\alpha - \overline{\omega}(q) - i\eta} - \overline{\theta}(p_m) \Big| \leq C \Big( \eta^{1/2} + \eta^{-1/2} |\alpha - \lambda^2 \Theta(\alpha) - e(p_m)| \Big) \,.$$

Therefore, using (B.5) and $\lambda^3 \ll \eta \ll \lambda^2$, we have

$$|\Omega(\alpha, p_m)| \leq C \lambda^2 \eta^{-1/2} \Big( 1 + \frac{\lambda^2 |\Theta(\alpha) - \theta(p_m)|}{|\alpha - \omega(p_m) + i\eta|} \Big) \leq C \lambda^2 \eta^{-1/2} \tag{5.15}$$

uniformly in $\alpha$ and $p_m$. $\square$

The proof shows that the cancellation between the gate and the $\vartheta$ is completely local in the graph. In particular, if we fix $L$ locations, maybe with multiplicity, between the elements of $V_{n,n'}$, and we consider all possible $2^L$ combinations of insertions of gates and $\vartheta$'s at these locations, then we gain a factor $\lambda^2\eta^{-1/2}$ from each location.

More precisely, let $\underline{\upsilon} \in \mathbf{N}^{\mathcal{V}_{n,n'}}$ be a given sequence of integers, $\upsilon_0, \upsilon_1, \ldots, \upsilon_{0^*}$, labelled by the elements of $\mathcal{V}_{n,n'}$. The number $\upsilon_j$ indicates how many gates or $\vartheta$'s are inserted between the $j$-th and $(j-1)$-th vertex. Let $|\underline{\upsilon}| := \sum_j \upsilon_j$ be the total number of insertions. A sequence $S \in \{g, \vartheta\}^{|\underline{\upsilon}|}$ encodes whether the insertion is gate or $\vartheta$.

Fix a sequence $\underline{\upsilon}$ and for any $S \in \{g, \vartheta\}^{|\underline{\upsilon}|}$ we define the extended set $\mathcal{V}_S$ consisting of $\mathcal{V}_{n,n'}$ and we insert extra single or double symbols for $\vartheta$ and gate indices (determined by $S$) at the locations given by $\underline{\upsilon}$. In the example above, we have $\underline{\upsilon} = (0, 0, \ldots, 0, 1, 0 \ldots 0)$ (i.e. $q_m = 1$, the rest is zero), $|\underline{\upsilon}| = 1$, and $\mathcal{V}'$ corresponds to $S = \{\vartheta\}$, while $\mathcal{V}''$ corresponds to $S = \{g\}$. Given a partition $\mathbf{P} \in \mathcal{P}_{n,n'}$, we also define the extended partition $\mathbf{P}_S$ on $\mathcal{V}_S$ by simply adding the single symbols as single lumps and the double symbols as paired lumps to $\mathbf{P}$.

**Lemma 5.5** *With the notations above, for any fixed $\mathbf{P} \in \mathcal{P}_{n,n'}$ and $\underline{\upsilon} \in \mathbf{N}^{\mathcal{V}_{n,n'}}$*

$$\Big| \sum_{S \in \{g,\vartheta\}^{|\underline{\upsilon}|}} V_{(*)}(\mathbf{P}_S) \Big| \le \Big( C\lambda^2\eta^{-1/2} \Big)^{|\underline{\upsilon}|} E_{(*)}(\mathbf{P}) \; . \tag{5.16}$$

*Proof.* Notice that the sum of the contributions of a gate and a $\vartheta$ inserted at the same place between $m$ and $m-1$ yields a factor of $\Omega(\alpha, p_m)$, while the same insertion between $\widetilde{m}$ and $\widetilde{m-1}$ yields a factor $\overline{\Omega(\beta, \tilde{p}_m)}$. These insertions are independent of each other, thanks to the summation over all possible $S$-combinations. So $\sum_S V(\mathbf{P}_S)$ is represented by an expression similar to (5.13), where a total factor

$$\prod_{m=1,\ldots,n,0^*} [\Omega(\alpha, p_m)]^{q_m} \prod_{m=\tilde{n}',\ldots,\tilde{1},0} [\overline{\Omega(\beta, \tilde{p}_m)}]^{q_m}$$

is inserted. The uniform estimate (5.15) for each $\Omega$ factor gives (5.16). $\square$.

Next we will apply Lemmas 5.4–5.5 to prove (4.37)–(4.39). The difficulty is that these estimates hold only if the gate remains isolated even after the lumping procedure, otherwise the gain comes from the artificial recollision introduced by the lump. Recall that the lumping procedure has two steps. The original partition $\mathbf{D}_0$ may lump non-trivially, yielding the derived partition $\mathbf{D}$, due to a few possible coincidences among the gate or recollision labels of $\psi$ and $\overline{\psi}$. Then the non-single elements of $\mathbf{D}$ (denoted by $\mathbf{D}^*$) lump into a coarser partition imposed by $A \in \mathcal{A}(\mathbf{D}^*)$ due to the connected graph formula.

We will estimate the value of individual graphs only. The number of terms in the summations in (4.37)–(4.39) is bounded by $O((c+4)^4) \le CK^4$. This extra factor $CK^4$ will be added to the factors gained in the cases discussed below to obtain (4.37)–(4.39).

### 5.3.2 Non-repetition graphs with a gate

To prove (4.37), we first note that the non-repetition rules in $\psi_{t,k}^{(1),nr}$ force $\mathbf{D}$ to be identical with $\mathbf{D}_0$ unless $h, h' = g$. In this latter case there is a gate both in the expansion of $\psi$ and $\overline{\psi}$, and either $\mathbf{D} = \mathbf{D}_0$ or $\mathbf{D}$ lumps the two gate-lumps in $\mathbf{D}_0$ together.

We first consider the case $\mathbf{D} \neq \mathbf{D}_0$. Then $\mathbf{D}$ has a lump consisting of all four gate indices, $\{w, w+1, \widetilde{w}, \widetilde{w+1}\}$. The corresponding Feynman graphs can be identified with certain nontrivial lumpings of non-repetition graphs on $I_{c+2} \cup \widetilde{I}_{c+2}$, where the indices $w$ and $w+1$ are lumped together. More precisely, for a given $\sigma \in \mathfrak{S}_c$, $w \in I_c$, $h = h' = g$, $\mathbf{A}' \in \mathcal{A}_c$,

$$\sum_{\mathbf{D} \neq \mathbf{D}_0} \sum_{\substack{\mathbf{A} \in \mathcal{A}(\mathbf{D}^*) \\ \mathbf{A} = \mathbf{A}'}} V(\mathbf{P}(\mathbf{A}, \mathbf{D})) c(\mathbf{A}) = \sum_{\substack{\mathbf{A} \in \mathcal{A}_{c+2}, \hat{\mathbf{A}} = \mathbf{A}' \\ w \equiv w+1 (mod\ \mathbf{A})}} V(\mathbf{A}, \widetilde{\sigma}) c(\mathbf{A}) \ , \tag{5.17}$$

where $\widetilde{\sigma} \in \mathfrak{S}_{c+2}$ is the natural extension of $\sigma$ where two new elements, $w, w+1$, are added to the base set $I_c$, the indices are shifted by the embedding map $s_w^h$ (Section 4.3) and $\sigma(w) = w$, $\widetilde{\sigma}(w+1) = \widetilde{w+1}$. The summation has at most $c+1$ terms and it expresses the choice of joining $w, w+1$ to one of the existing lumps in $\mathbf{A}'$ or keeping the lump $\{w, w+1\}$ separate in $\mathbf{A}$. Because of lumping $w$ and $w+1$, the partition $\mathbf{A}$ is non-trivial, and $q(\mathbf{A}, \widetilde{\sigma}) \geq 1$ (see (4.26)). After estimating $|V(\cdots)|$ by $E(\cdots)$, each term on the right hand is estimated by the bound

$$\sup_{\mathbf{u}} E_{(*)g}(\mathbf{A}, \sigma, \mathbf{u}) \leq C |\log \lambda|^2 \Big( \lambda^{\frac{1}{3} - (\frac{17}{3} d + \frac{3}{2})\kappa - O(\delta)} \Big)^{q(\mathbf{A}, \sigma)} \tag{5.18}$$

that holds whenever $\sigma \in \mathfrak{S}_k$, $\mathbf{A} \in \mathcal{A}_k$ and $\kappa \leq \frac{2}{34d+9}$ (see (9.4) from [10]).

The estimate (5.18), combined with the combinatorial bound on $c(\mathbf{A})$ and on the summation (Lemma 4.5), gives (4.37) in the case when $\mathbf{D} \neq \mathbf{D}_0$.

Now we focus on the case $\mathbf{D} = \mathbf{D}_0$. If at least one of the gate-lumps, $\{w, w+1\}$ or $\{\widetilde{w}, \widetilde{w+1}\}$, do not remain isolated in $\mathbf{A}$, then we can repeat the argument above since $\mathbf{P}(\mathbf{A}, \mathbf{D})$ has a non-trivial lump of size at least 4.

Finally, we can assume that the gate lumps remain isolated in $\mathbf{A}$. We fix $\mathbf{A}'$ and consider the sum of four terms corresponding to $h, h' \in \{g, \theta\}$. The partition $\mathbf{A}$ is defined by adding the gate lump(s) to $\mathbf{A}'$. Note that $c(\mathbf{A})$ is the same for all these four cases since replacing a $\theta$ index with an isolated gate lump adds only a single lump to $\mathbf{A}$. For these partitions, we apply Lemma 5.5 (with the choice $\upsilon_w = \upsilon_{\widetilde{w}}$, all other $\upsilon$'s are zero) to obtain a factor $(C\lambda^{1-\frac{\kappa}{2}})^2$. The remaining non-repetition graphs bounded by $O(|\log \lambda|^2)$ by using Proposition 9.2 from [10] with $q = 0$. This completes the proof of (4.37). $\square$

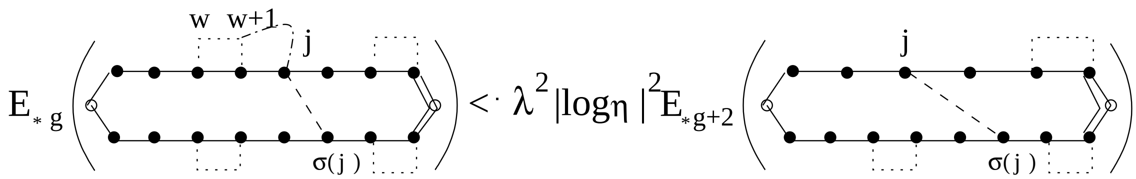


Figure 11: Case 1. Removal of a gate lumped to an adjacent core index

### 5.3.3 Last gate

First we consider the case when the lumps $\{w, w+1\} \subset I_n$ and $\{\widetilde{w}, \widetilde{w+1}\} \subset \widetilde{I}_{n'}$ of the two first gates remain isolated in $\mathbf{P} = \mathbf{P}(\mathbf{A}, \mathbf{D})$. As in the previous section, by applying Lemma 5.5, we can sum up the two times two possibilities for the first components of the code $h$ and $h'$, i.e. sum up four Feynman diagrams that differ only by the choice of gate or $\theta$ at the $w$ or $\widetilde{w}$ position. We collect $(\lambda^{1-\kappa/2})^2$. The remaining two gates will be isolated from the rest in $\mathbf{P}$ by Operation I, then half of each gate is removed by Operation III, collecting $\Lambda^2\lambda^2|\log\lambda|^2$. Finally, (C.2) can be used to remove the remaining two halves of the gates, collecting $\lambda^2$. By (5.1), the untruncated $E$-values of the remaining graph are bounded by $O(|\log\lambda|^2)$. The total estimate is $C\lambda^{6-(4d+1)\kappa-O(\delta)}$.

If only one of the lumps, $\{w, w+1\}$ or $\{\widetilde{w}, \widetilde{w+1}\}$, remains isolated in $\mathbf{P}$, we can still apply Lemma 5.4 to obtain a cancellation of order $\lambda^{1-\kappa/2}$ from adding up the $V$-values of those pairs of graphs that differ only by changing this gate to $\theta$. Note that the value $c(\mathbf{A})$ is again the same for these two graph.

Next we consider those lumps among $\{w, w+1\}$ and $\{\widetilde{w}, \widetilde{w+1}\}$ that do not remain isolated in $\mathbf{P}$. By breaking up lumps via Operation I, we can ensure that every such gate is either lumped exactly with one other gate or with a core index pair. For definiteness, let $\{w, w+1\}$ from $I_n$ be such a gate. We also isolate all other gates from the rest. To do that, Operation I is used at most four times at the total expense of $\Lambda^4$. We distinguish three cases:

*Case 1.* The gate $\{w, w+1\}$ is lumped with a core index-pair $(j, \widehat{\sigma}(j))$, where $\widehat{\sigma}$ is the natural extension of $\sigma$ from $I_c$ to $I_n$. If $j$ is next to $w$ or $w+1$, say $j = w+2$ (Fig. 11), then both vertices of the gate can be removed by Operation III since the momenta between $(w-1, w)$ and $(w, w+1)$ do not appear in any delta function, and we gain $\lambda^2|\log\eta|^2$ from this gate. We can now remove the two gates at the end (using Operation III and (C.2) as above), collect an additional $\lambda^4|\log\lambda|^2$. The remaining gate at $\{\widetilde{w}, \widetilde{w+1}\}$ can be removed by Operation IV at the expense of $\lambda^{-\kappa}|\log\lambda|$. By Proposition 9.2 from [10], the remaining graphs are bounded by $O(|\log\lambda|^2)$. We thus collect $C\lambda^{6-(8d+1)\kappa-O(\delta)}$.

If $j$ is not next to $w$ or $w+1$ (Fig. 12), then we remove $w+1$ by Operation III, break

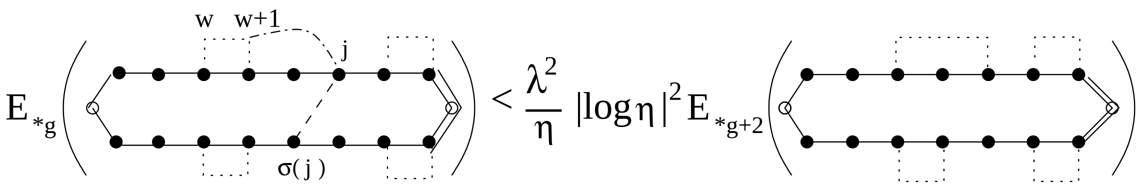


Figure 12: Case 1. Removal of a gate lumped to a non-adjacent core index

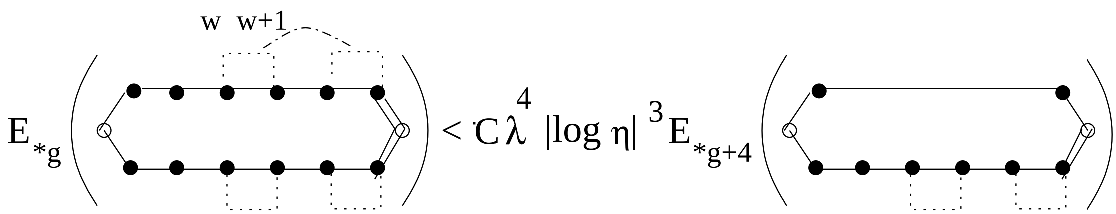


Figure 13: Case 2. Removal of two adjacent lumped gates

up the lump into $\{j, w\}$ and $\{\widehat{\sigma}(j)\}$ by Operation I and remove the single lump $\{\widehat{\sigma}(j)\}$ by Operation II. The total price for these steps is $\Lambda\lambda^2\eta^{-1}|\log\eta|^2$. We can now again remove the two gates at the end, collect $\lambda^4|\log\lambda|^2$ and we end up with a graph with a one sided recollision, so (5.6) from Proposition 5.3 applies. We collect $C\lambda^{6-(14d+2)\kappa-O(\delta)}$ in this case.

*Case 2.* The gate $\{w, w+1\}$ is lumped with the other gate in $I_n$, i.e. with $\{n-1, n\}$. If $w+1$ and $n-1$ are neighbors (Fig. 13), then we can remove three vertices $n-2 = w+1, n-1, n$ by Operation III. We collect $\lambda^3|\log\eta|^3$. On the $\widetilde{I}_{n'}$ side, we remove the gate that is not adjacent to $0^*$ by Operation IV at the expense of $\lambda^{-\kappa}|\log\lambda|$ as above. The other gate is adjacent to $0^*$; first we remove its leg not adjacent to $0^*$ by Operation III, collecting $\lambda|\log\lambda|$. Finally, we remove the two remaining vertices, originally with indices $n-3$ and $n'$ that now form single lumps and they are both adjacent to $0^*$. By using (C.2), we gain a factor $\lambda^2$. Altogether we thus collect $C\lambda^{6-(8d+1)\kappa-O(\delta)}$.

If $w+1$ and $n-1$ are not neighbors (Fig. 14), then we remove $w+1$ and $n$, gaining $\lambda^2|\log\eta|^2$ and the remaining partition again has a one-sided recollision. The other gate adjacent to $0^*$ can be removed by collecting $\lambda^2|\log\lambda|^2$, the remaining gate collects $\lambda^{-\kappa}|\log\lambda|$, and finally we use (5.6). The result is again $\lambda^{6-(12d+1)\kappa-O(\delta)}$.

*Case 3.* Finally, if neither $\{w, w+1\}$ nor $\{\widetilde{w}, \widetilde{w+1}\}$ falls into Case 1 or 2, then each of them either remains isolated and collects $\lambda^{1-\kappa/2}$ from Lemma 5.4 or is lumped with another gate on the "opposite side" (Fig. 15). For definiteness, assume $\{w, w+1\}$ is lumped with $\{\widetilde{\ell}, \widetilde{\ell+1}\}$ in $\widetilde{I}_{n'}$, where $\widetilde{\ell}$ is either $\widetilde{w}$ or $\widetilde{n'-1}$. Then we simply remove half

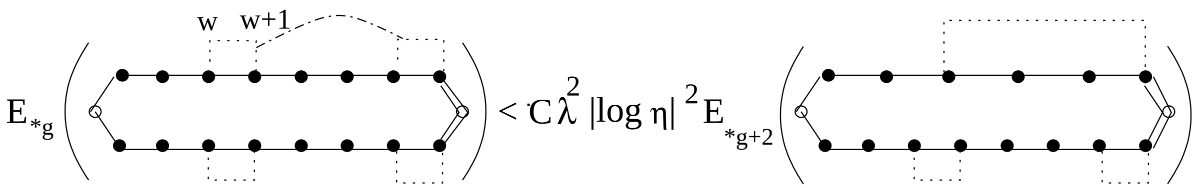


Figure 14: Case 2. Removal of half of each gates lumped on the same side

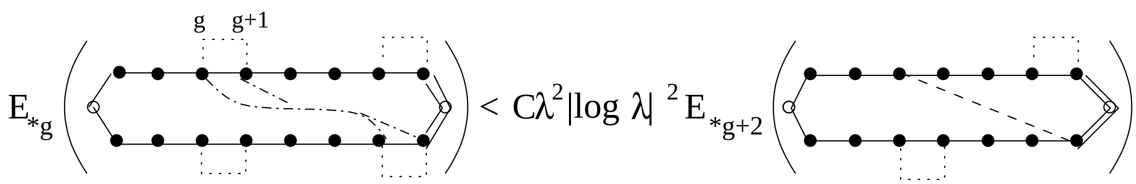


Figure 15: Case 3. Removal of two opposite lumped gates

of each gates, say $w+1$ and $\widetilde{\ell+1}$ using Operation III, gain $\lambda^2|\log\eta|^2$ and we extend the set of core indices to include $w$ and extend the permutation $\sigma$ by adding $\sigma(w)=\ell$. Therefore we effectively gained $\lambda|\log\eta|$ from each such gate. Finally, after having gained either $\lambda^{1-\kappa/2}$ or $\lambda|\log\lambda|$ from each gate, we gain $\lambda^2|\log\lambda|^2$ from the remaining truncated graph (5.1)–(5.2) and we thus collect at least $C\lambda^{6-(8d+1)\kappa-O(\delta)}$. This completes the proof of (4.38). $\square$.

### 5.3.4 Nest

The procedure is very similar to the analysis of the last gate, so we just outline the steps. We point out that the main reason why the nested graphs are small is the cancellation between the gate and $\theta$ inside the nest. This is a different mechanism than the one used in [7].

First we consider the cases when both nests are independent, i.e. no part of the nest in $I_n$ is lumped with any part of the nest in $\widetilde{I}_{n'}$. We will show that one can gain at least $\lambda^{3-(2d+2)\kappa-O(\delta)}$ from each such nest. The total gain from both nests is then $\lambda^{6-(4d+4)\kappa-O(\delta)}$.

Consider the gate $\{n-2,n-1\}$ inside the nest. If this gate remains an isolated lump in $\mathbf{P}$, then it cancels the same graph with $\theta$ up to order $\lambda^{1-\kappa/2}$ by Lemma 5.4. After this cancellation, the outer shell of the nest, $\{n-3,n\}$, becomes a gate in the reduced graph that is adjacent with $0^*$. After separating it from the rest by Operation I, if necessary, its removal yields $\lambda^2$ because of the truncation. Therefore we gain at least

$\Lambda\lambda^{3-\kappa/2} \le \lambda^{3-(2d+\frac{1}{2})\kappa - O(\delta)}$ from this nest.

If the gate $\{n-2, n-1\}$ is not isolated in $\mathbf{P}$, then we again distinguish a two cases.

If $\{n-2, n-1\}$ is lumped with a core index $j < n-3$ but not with its outer shell, $\{n-3, n\}$, then we can always create a one-sided recollision in such a way that $n-2$ will be paired with $\sigma(j)$ in the extended permutation, while $n-1$ is removed (Operation III, gain $\lambda|\log\lambda|$) and $j$ is removed (Operation II, lose $\lambda^{-1-\kappa}$). The net result is $\lambda^{-\kappa}|\log\lambda|$ and the outer shell of the nest, $\{n-3, n\}$, becomes a genuine recollision. We may have to separate this from the rest of the graph by an Operation I before applying (5.7). This will effectively give $\Lambda\lambda^{3-2\kappa}\zeta^{4d}|\log\lambda|^{O(1)} \le \lambda^{3-(6d+2)\kappa - O(\delta)}$ for this nest. In this calculation we used only $\lambda^{3-\kappa}$ from (5.7), the additional $\lambda$ is due to the truncation on the "other side" and will be counted there.

If the gate $\{n-2, n-1\}$ is lumped with its outer shell, then the momenta between $(n-3, n-2)$, $(n-2, n-1)$ and $(n-1, n)$ can be freely integrated, we can remove the nest completely and collect $\Lambda\lambda^4 \le C\lambda^{4-2d\kappa}$ from this nest. We may have to use Operation I once to separate the nest from the rest of the graph.

So far we have treated the cases when the two nests are independent. In the remaining cases some parts of the nests are lumped with each other.

If the gate $\{n-2, n-1\}$ is lumped with the other gate $\{\widetilde{n-2}, \widetilde{n-1}\}$, then half of each gate is removed, say $n-1, \widetilde{n-1}$ by Operation III (gain $\lambda^2|\log\lambda|^2$), the other halves are separated from the rest of the graph (Operation I) and then connected by an extending the permutation $\sigma(n-2) = n-2$ (i.e. we include $n-2$ among the core indices). The resulting graph has two recollisions, so (5.5) applies and the total size is $\Lambda\lambda^{8-3\kappa}\zeta^{4d}|\log\lambda|^{O(1)}$.

Finally, if $\{n-2, n-1\}$ is lumped with the outer shell $\{\widetilde{n-3}, \widetilde{n}\}$ of the other nest, then we isolate and remove the gate $\{\widetilde{n-2}, \widetilde{n-1}\}$ inside the other nest (price: $\Lambda\lambda^{-\kappa}|\log\lambda|$), remove $\widetilde{n}$ and $n-1$ each as one half of a gate, gaining $\lambda^2$ and the remaining graph is a one-sided truncated recollision graph after a possibly isolating the recollision lump $\{n-3, n\}$ from the rest. Thus (5.7) gives $\Lambda\lambda^{4-\kappa}\zeta^{4d}|\log\lambda|^{O(1)}$, after a possible application of Operation I. The total size is $O(\lambda^{6-(8d+2)\kappa - O(\delta)})$. Collecting the various cases, we obtain (4.39). $\square$

# 6 Convergence of the ladder diagrams to the heat equation

We start the proof of Theorem 2.4 by noticing that

$$W_\lambda(t, k, \mathcal{O}) = \int V^\circ_{\varepsilon\xi}\big(\mathbf{A}_0, \widehat{\mathcal{O}}(\xi, \cdot)\big)\mathrm{d}\xi, \qquad k \ge 1$$

with $\mathbf{A}_0$ being the trivial partition on $I_k$, where we chose the function $Q(v)$ in the definition of $V^\circ$ to be $\xi$-dependent, namely $Q(v) = Q_\xi(v) := \widehat{\mathcal{O}}(\xi, v)$ (see (4.17) for the definition of $V^\circ$). First we note that the d$\xi$ integration can be restricted to the regime $\{|\xi| \le \lambda^{-\delta}\}$ with a negligible error (even after summation over $k$):

$$\sum_{1\le k<K} W_\lambda(t,k,\mathcal{O}) = \sum_{1\le k<K} \Xi_k^\circ + o(1) , \qquad \Xi_k^\circ := \int^* V_{\varepsilon\xi}^\circ\big(\mathbf{A}_0, \widehat{\mathcal{O}}(\xi,\cdot)\big)\mathrm{d}\xi , \tag{6.1}$$

where use the notation

$$\int^* (\cdots)\mathrm{d}\xi := \int (\cdots)\mathbf{1}(|\xi| \le \lambda^{-\delta})\mathrm{d}\xi .$$

To see (6.1), we first recall that replacing $V^\circ(\cdots)$ with $V(\cdots)$ yields a negligible error even after summing up for all $k$ (Lemma 7.1 of [10]). The linearity of the estimate in $\|Q_\xi\|_\infty = \sup_v |\widehat{\mathcal{O}}(\xi,v)|$ guarantees the integrability in $\xi$ since $\mathcal{O}$ is a Schwarz function. We then use the estimate

$$\Big|V_{\varepsilon\xi}\big(\mathbf{A}_0, \widehat{\mathcal{O}}(\xi,\cdot)\big)\Big| \le \|Q_\xi\|_\infty \sup_{\xi,\mathbf{u}} E(\sigma = id, \mathbf{u})$$

and the uniform bound (5.1) and finally we conclude (6.1) by the arbitrarily fast decay of $\|Q_\xi\|_\infty$ in $\xi$.

From the definition of $\Xi_k^\circ$, we have

$$\begin{aligned}\Xi_k^\circ =& \lambda^{2k} \iint_{\mathbb{R}} \frac{\mathrm{d}\alpha\mathrm{d}\beta}{(2\pi)^2}\, e^{it(\alpha-\beta)+2t\eta} \int^* \mathrm{d}\xi \int \prod_{j=1}^{k+1} \mathrm{d}\mu(v_j)\; \widehat{\mathcal{O}}(\xi, v_{k+1})\overline{\widehat{W}_0}(\varepsilon\xi, v_1)\\ &\times \prod_{j=1}^{k+1} \Big[\overline{R_\eta\Big(\alpha, v_j + \frac{\varepsilon\xi}{2}\Big)} R_\eta\Big(\beta, v_j - \frac{\varepsilon\xi}{2}\Big)|\widehat{B}(v_j - v_{j+1})|^2\Big] .\end{aligned} \tag{6.2}$$

To simplify the notation, in (6.2) we followed the convention, that $|\widehat{B}(v_j - v_{j+1})| = 1$ for $j = k+1$ because of the non-existence of $v_{k+2}$. Similar convention will be followed later, also for $|\widehat{B}(v_{j-1} - v_j)|^2 = 1$ if $j = 1$. Note also that the measure d$v_j$ has been changed to d$\mu(v_j)$ by using the support properties of $\widehat{B}$ and $\psi_0$ (2.1).

The estimates of the error terms were performed with the choice $\eta = \lambda^{2+\kappa}$. However, $\Xi_k^\circ$, given by (6.2), is clearly *independent* of $\eta$; this follows from the $K$-identity (formula (6.2) in [10]). Therefore we can change the value of $\eta$ to $\eta := \lambda^{2+4\kappa}$ for the rest of this calculation and we define

$$R(\alpha, v) := R_\eta(\alpha, v) , \qquad \text{with} \quad \eta := \lambda^{2+4\kappa} .$$

We also recall, that the restriction of the d$\alpha$d$\beta$ integration in (6.2) to any set that contains $\{\alpha,\beta\;:\;|\alpha|,|\beta|\le Y=\lambda^{-100}\}$ results in negligible errors, even after the summation over $k$ (Lemma 7.1 of [10]). We will consider the set $D:=\{(\alpha,\beta)\;:\;|\alpha+\beta|\le 2Y, |\alpha-\beta|\le 2Y\}$. We denote by $\Xi_k$ the version of $\Xi_k^\circ$ given by formula (6.2) with the d$\alpha$d$\beta$ integrals restricted to $D$,

$$\Xi_k := \lambda^{2k}\iint_D \frac{\mathrm{d}\alpha\mathrm{d}\beta}{(2\pi)^2}\Big[\text{Integrand from (6.2)}\Big] ,$$

then

$$\sum_{1\le k\le K}|\Xi_k^\circ-\Xi_k| = o(1) .$$

We also remind the reader that this argument literally does not apply to the trivial $k=0$ case, when the d$\alpha$ d$\beta$ integral in (6.2) gives free evolutions and this term should be computed directly:

$$\Xi_0 := \int^* \mathrm{d}\xi\mathrm{d}v\; e^{it\varepsilon v\cdot\xi}\; e^{2t\lambda^2 \mathrm{Im}\,\theta(v)}\; \widehat{\mathcal{O}}(\xi,v)\overline{\widehat{W}_0}(\varepsilon\xi,v)+o(1) , \tag{6.3}$$

where the error term comes from the error term in $\overline{\theta}(v+\varepsilon\xi/2)-\theta(v-\varepsilon\xi/2)=2i\mathcal{I}(v)+O(\varepsilon\xi)$. By using $t\lambda^2\to\infty$, the bound

$$\mathrm{Im}\;\theta(v)\le -c_1\min\{|p|^{d-2},|p|^{-1}\} ,$$

(from Lemma 3.2 of [10]) and the decay of the observable, one easily obtains that $|\Xi_0| = o(1)$ anyway.

To evaluate the integral (6.2), we need the following crucial technical lemma which is proven in the Appendix.

**Lemma 6.1** *Let $\kappa<1/8$, define $\gamma:=(\alpha+\beta)/2$ and let $\eta$ satisfy $\lambda^{2+4\kappa}\le\eta\le\lambda^{2+\kappa}$. Then for $|r|\le\lambda^{2+\kappa/4}$ we have,*

$$\begin{aligned}\Omega := &\int \frac{\lambda^2 f(p)}{\Big(\alpha-\bar\omega(p-r)-i\eta\Big)\Big(\beta-\omega(p+r)+i\eta\Big)}\,\mathrm{d}p\\ &= -2\pi i\lambda^2\int\frac{f(p)\;\delta(e(p)-\gamma)}{(\alpha-\beta)+2p\cdot r-2i[\lambda^2\mathcal{I}(\gamma)+\eta]}\,\mathrm{d}p+O(\lambda^{1/2-4\kappa})\|f\|_{4d,1} .\end{aligned} \tag{6.4}$$

Now we compute $\Xi_k$ by applying Lemma 6.1. Denote $a:=(\alpha+\beta)/2$ and $b:=\lambda^{-2}(\alpha-\beta)$. By using $\varepsilon=\lambda^{2+\kappa/2}$, $\eta=\lambda^{2+4\kappa}$, we have

$$\begin{aligned}&\lambda^2\int \mathrm{d}v\;\Upsilon(\xi,v)\;\overline{R\Big(\alpha,v+\frac{\varepsilon\xi}{2}\Big)}R\Big(\beta,v-\frac{\varepsilon\xi}{2}\Big)\\ &\qquad=\int\frac{-2\pi i\Upsilon(\xi,v)\;\delta(e(v)-a)}{b+\lambda^{\kappa/2}v\cdot\xi-2i[\mathcal{I}(a)+\lambda^{4\kappa}]}\,\mathrm{d}v+O(\lambda^{1/2-4\kappa})\|\Upsilon\|_{4d,1} .\end{aligned} \tag{6.5}$$

In the applications, $\Upsilon(\xi, v)$ will always be supported on $|v| \leq \zeta$, therefore the measure $\mathrm{d}v$ can be freely changed to $\mathrm{d}\mu(v)$.

We now replace the product of $k+1$ factors in the restricted version of (6.2) one by one. We need a $\lambda^2$ factor for each application of (6.5), thus we need $\lambda^{2k+2}$ in (6.2). But (6.2) contains only $\lambda^{2k}$, the missing $\lambda^2$ comes from the change of variables $\mathrm{d}\alpha \mathrm{d}\beta = \lambda^2 \mathrm{d}a\mathrm{d}b$. The domain of integration, $(\alpha, \beta) \in D$, is replaced by the domain $D^* := \{(a, b) : |a| \leq Y, |b| \leq 2\lambda^{-2} Y\}$.

For any $\ell = 1, 2, \ldots k+1$, we introduce the notation

$$
\begin{aligned}
\mathcal{F}_{k,\ell} := \int^* \mathrm{d}\xi \int_{D^*} \frac{\mathrm{d}a\mathrm{d}b}{(2\pi)^2} \Big( \prod_{\substack{j=1 \\ j\neq \ell}}^{k+1} \int \mathrm{d}\mu(v_j) \Big) \prod_{j=1}^{\ell-2} \left( \left| \frac{2\pi i \; F_j(\xi, v_j) |\widehat{B}(v_j - v_{j+1})|^2 \; \delta(e(v_j) - a)}{b + \lambda^{\kappa/2} v_j \cdot \xi - 2i[\mathcal{I}(a) + \lambda^{4\kappa}]} \right| \right) \\
\times \left| \frac{2\pi i F_\ell(\xi, v_{\ell-1}) \; \delta(e(v_{\ell-1}) - a)}{b + \lambda^{\kappa/2} v_{\ell-1} \cdot \xi - 2i[\mathcal{I}(a) + \lambda^{4\kappa}]} \right| \\
\times \left| \int \mathrm{d}\mu(v_\ell) \Upsilon_\ell(v_{\ell-1}, v_\ell, v_{\ell+1}) \left[ \lambda^2 \overline{R\Big(\alpha, v_\ell + \frac{\varepsilon\xi}{2}\Big)} R\Big(\beta, v_\ell - \frac{\varepsilon\xi}{2}\Big) \right. \right. \\
\left. \left. - \frac{-2\pi i \; \delta(e(v_\ell) - a)}{b + \lambda^{\kappa/2} v_\ell \cdot \xi - 2i[\mathcal{I}(a) + \lambda^{4\kappa}]} \right] \right| \\
\times \prod_{j=\ell+1}^{k} \left( \lambda^2 \left| R\Big(\alpha, v_j + \frac{\varepsilon\xi}{2}\Big) R\Big(\beta, v_j - \frac{\varepsilon\xi}{2}\Big) F_j(\xi, v_j) \; |\widehat{B}(v_j - v_{j+1})|^2 \right| \right) \\
\times \lambda^2 \left| R\Big(\alpha, v_{k+1} + \frac{\varepsilon\xi}{2}\Big) R\Big(\beta, v_{k+1} - \frac{\varepsilon\xi}{2}\Big) F_{k+1}(\xi, v_{k+1}) \right|
\end{aligned}
\tag{6.6}
$$

with

$$
\Upsilon_\ell(v_{\ell-1}, v_\ell, v_{\ell+1}) := \begin{cases} \widehat{W_0}(\varepsilon\xi, v_1) \, |\widehat{B}(v_1 - v_2)|^2 & \text{for} \quad \ell = 1 \\ |\widehat{B}(v_{\ell-1} - v_\ell)|^2 |\widehat{B}(v_\ell - v_{\ell+1})|^2 & \text{for} \quad 2 \leq \ell \leq k \\ |\widehat{\mathcal{O}}(\xi, v_{k+1})| \, |\widehat{B}(v_1 - v_2)|^2 & \text{for} \quad \ell = k+1 \end{cases}
$$

and

$$
F_j(\xi, v) := \begin{cases} \overline{\widehat{W}}_0(\varepsilon\xi, v) & \text{for} \quad j = 1 \\ 1 & \text{for} \quad 2 \leq j \leq k \\ \widehat{\mathcal{O}}(\xi, v) & \text{for} \quad j = k+1 \, . \end{cases}
$$

The formula (6.6) is literally valid for $2 \leq \ell \leq k$. For $\ell = 1$ the first product and the factor in the second line are absent, for $\ell = k+1$ the factors in the last two lines are absent. We also recall the convention made after (6.2) about the intepretation of $|\widehat{B}(v_j - v_{j+1})|$ for $j = k+1$.

With these notations and by introducing $\tau := \lambda^2 t = \lambda^{-\kappa} T$, and $W(k) := W_\lambda(t,k,\mathcal{O})$, we obtain the following telescopic estimate from (6.2)

$$\Big| \sum_{k<K} W(k) - \sum_{k<K} \int^* \mathrm{d}\xi \int_{D^*} \frac{\mathrm{d}a\mathrm{d}b}{(2\pi)^2}\, e^{i\tau b + 2t\eta}$$
$$\times \left( \prod_{j=1}^{k+1} \int \frac{-2\pi i F^{(j)}(\xi, v_j)\, \delta(e(v_j)-a)}{b + \lambda^{\kappa/2} v_j\cdot\xi - 2i[\mathcal{I}(a)+\lambda^{4\kappa}]} |\widehat{B}(v_j - v_{j+1})|^2 \mathrm{d}\mu(v_j) \right) \Big|$$
$$\le \sum_{k<K} \sum_{\ell=1}^{k+1} \mathcal{F}_{k,\ell} + o(1)\ . \tag{6.7}$$

Now we explain how to estimate $\mathcal{F}_{k,\ell}$ for the general case $(2\le \ell\le k)$, the modifications for the two extrema are straighforward.

First we estimate $\widehat{\widehat{W}}_0$ by supremum norm and estimate all denominators in the first two lines by their imaginary part:

$$\Big| \frac{1}{b + \lambda^{\kappa/2} v_j\cdot\xi - 2i[\mathcal{I}(a)+\lambda^{4\kappa}]} \Big| \le \frac{1}{2\mathcal{I}(a)}\ . \tag{6.8}$$

Then the $v_1, v_2, \ldots, v_{\ell-2}$ variables are integrated out in this order, by using (2.4), yielding a total factor 1 from the product in the first line of (6.6). By recalling (1.9) and the estimate

$$\mathcal{I}(a) = -\mathrm{Im}\, \Theta(a) \ge c_1 \min\{|e|^{\frac{d}{2}-1}, e^{-1/2}\}$$

from Lemma 3.2 of [10], the integral of $v_{\ell-1}$ is estimated trivially by

$$\frac{1}{2\mathcal{I}(a)} \int \delta(e(v_{\ell-1}) - a)\, \mathrm{d}\mu(v_{\ell-1}) \le \frac{Ca^{1/2}}{\mathcal{I}(a)} \le \langle a\rangle\ . \tag{6.9}$$

This estimate is used if $a\le \zeta^2/2$, otherwise the integral is zero by the support of $\mathrm{d}\mu$, so we obtain a factor $O(\lambda^{-2\kappa - O(\delta)})$. In the regime $|b|\ge \lambda^{-\kappa}$, we have $|b-\lambda^{\kappa/2}v_{\ell-1}\cdot\xi| \ge |b|/2$ using $|v_{\ell-1}|\le\zeta$ and $|\xi|\le\lambda^{-\delta}$. The estimate (6.8) can thus be changed to $2|b|^{-1}$, improving estimate (6.9) to $\le Ca^{1/2}/|b| \le C\zeta/|b|$.

The integral $\mathrm{d}\mu(v_\ell)$ in (6.6) is estimated by

$$O(\lambda^{1/2-4\kappa}) \sup_{v_{\ell-1}, v_{\ell+1}} \big\| \Upsilon_\ell(v_{\ell-1}, \,\cdot\,, v_{\ell+1}) \big\|_{4d,1} \le O(\lambda^{1/2-4\kappa}\zeta^{4d}) = O(\lambda^{1/2-(4d+4)\kappa - O(\delta)}) \tag{6.10}$$

by using (6.5) and the fact that all $v_j$ variables satisfy $|v_j|\le\zeta$. For $\ell=1$ and $\ell=k+1$ we also used that the initial data and the observable are Schwarz functions.

For the $d\mu(v_j)$, $j = \ell+1, \ell+2, \ldots, k$, integrals we separate the resolvents by Schwarz inequality,

$$\prod_{j=\ell+1}^{k} \big|R(\alpha, v_j + \ldots)R(\beta, v_j - \ldots)\big| \leq \prod_{j=\ell+1}^{k} \big|R(\alpha, v_j + \ldots)\big|^2 + \prod_{j=\ell+1}^{k} \big|R(\beta, v_j - \ldots)\big|^2 ,$$

and we use the successive integration scheme (see Section 10.1.2 of [10]) to collect a constant factor.

Before we integrate out the last momentum variable, $v_{k+1}$, we perform the $da\,db$ integration. We can change back the $a, b$ variables to $\alpha, \beta$, we perform $d\alpha\, d\beta$ integrals to collect a $C|\log\lambda|^2$ factor since $D \subset \{|\alpha|, |\beta| \leq 2Y\}$. This argument applies unless $\ell = k+1$ and the last line in (6.6) is absent. In this case, however, $\ell \geq 2$ (the $k = 0$ case is treated separately, see (2.23)), and then the denominator with $v_{\ell-1}$ in the second line of (6.6) is present. We use the $a, b$ variables. Recall that $\delta(e(v_{\ell-1}) - a)$ restricts the domain of the $da$ integration to $|a| \leq \zeta^2/2$, giving a contribution $O(\zeta^2)$. The domain of the $b$ integral is larger, $|b| \leq 2\lambda^{-2}Y$, but in the regime $|b| \geq \lambda^{-\kappa}$ we have collected an additional $|b|^{-1}$ factor in (6.9), thus the $db$ integration contributes at most with a factor $O(\lambda^{-\kappa})$.

Finally we integrate $v_{k+1}$ by using the integrability of $F_{k+1} = \mathcal{O}$ and the $\xi$ integral gives a factor $O(\lambda^{-3\delta})$. By collecting these estimates, we arrive at

$$\sum_{1\leq k<K} \sum_{\ell=1}^{k+1} \mathcal{F}_{k,\ell} \leq C\lambda^{1/2-(4d+9)\kappa-O(\delta)}$$

and that is negligible, since $\kappa < 1/(8d+18)$.

Now we focus on the main term on the left hand side of (6.7). First we extend the $db$ integration from $|b| \leq 2\lambda^{-2}Y$ to $\mathbb{R}$. It is easy to see that the error is negligible; all denominators can be bounded by $|b|/2$ and the result from the $db$ integral in the region $|b| \geq 2\lambda^{-2}Y$,

$$\int_{|b|\geq 2\lambda^{-2}Y} \frac{db}{|b|^{k+1}} \leq \lambda^{2k} ,$$

is negligible even after multiplying the $C^k$ from the $dv_j$ integrals. We can also extend the $da$ integration from $[-Y, Y]$ to $\mathbb{R}$, since, due to the factor $\delta(e(v_j) - a)$ and the cutoff in $v_j$, the integrand is zero for $|a| \geq Y$.

Now we write

$$\frac{-i}{b + \lambda^{\kappa/2} v_j \cdot \xi - 2i[\mathcal{I}(a) + \lambda^{4\kappa}]} = \int_0^\infty e^{-i\tau_j(b+\lambda^{\kappa/2}\ldots)} d\tau_j$$

and perform the d$b$ integration. We obtain

$$\sum_{k<K} W(k) = \sum_{k<K} \int^* \mathrm{d}\xi \int_{\mathbb{R}} \frac{\mathrm{d}a}{2\pi}\, e^{-2\tau\mathcal{I}(a)-\tau\lambda^{4\kappa}} \Big(\prod_{j=1}^{k+1}\int_0^\infty \mathrm{d}t_j\Big)\delta\Big(\tau-\sum_{j=1}^{k+1}\tau_j\Big)\; e^{-i\lambda^{\kappa/2}(\sum\tau_j v_j)\cdot\xi}$$
$$\times\Big(\prod_{j=1}^{k+1}\int \mathrm{d}\mu(v_j)\; 2\pi|\widehat{B}(v_j-v_{j+1})|^2\delta(e(v_j)-a)\Big)\widehat{\mathcal{O}}(\xi, v_{k+1})\overline{\widehat{W}}_0(\varepsilon\xi, v_1) + o(1)\;.$$

We now replace d$\mu(v_j)$ with d$v_j$ and remove the cutoff in $\xi$ to perform the Fourier transform. We also replace $W_0$ with $F_0$ and remove the $k<K$ cutoff from the summation:

$$\sum_{k<K} W(k) = \sum_{k=0}^{\infty}\int \mathrm{d}X \int_{\mathbb{R}} \mathrm{d}a\; e^{-2\tau\mathcal{I}(a)}\Big(\prod_{j=1}^{k+1}\int_0^\infty \mathrm{d}\tau_j\Big)\delta\Big(\tau-\sum_{j=1}^{k+1}\tau_j\Big)$$
$$\times\int \mathrm{d}v_1\,\delta(e(v_1)-a)\left(\prod_{j=2}^{k+1}\int \mathrm{d}v_j\; 2\pi|\widehat{B}(v_j-v_{j-1})|^2\delta(e(v_j)-a)\right)$$
$$\times\,\mathcal{O}(X, v_{k+1})F_0\Big(X-(2\pi)^{-1}\lambda^{\kappa/2}(\sum_j \tau_j v_j),\; v_1\Big) + o(1) \tag{6.11}$$

with initial data

$$F_0(X, v) := \delta(X)|\widehat{\psi}_0(v)|^2\;.$$

The replacement of d$\mu(v_j)$ with d$v_j$ is justified since $\widehat{\psi}_0(v_1)$ is compactly supported and thus all other $v_j$'s are restricted to a compact energy range by the delta functions $\prod_j \delta(e(v_j)-a)$. The removal of the $\xi$-cutoff is allowed since the integrand of the d$\xi$-integral can be majorized by

$$\sup_v|\widehat{\mathcal{O}}(\xi,v)|\int_{\mathbb{R}}\frac{\mathrm{d}a}{2\pi}\, e^{-2t\mathcal{I}(a)}\sum_{k\le K}\frac{\big[2\mathcal{I}(a)t\big]^k}{k!}\int \mathrm{d}v_1\;\delta(e(v_1)-a)\overline{\widehat{W}}_0(\varepsilon\xi, v_1)\le \sup_v|\widehat{\mathcal{O}}(\xi,v)| \tag{6.12}$$

whose integral vanishes in the regime $|\xi|\ge\lambda^{-\delta}$ due to the assumptions on $\widehat{\mathcal{O}}$. Here we used (2.4) to perform the $v_j$ integrations successively and the time integration yielded $\tau^k/k!$. The replacement of $\overline{\widehat{W}}_0(\varepsilon\xi, v)$ with $|\widehat{\psi}_0(v)|^2$ comes from the uniformly integrable bound (6.12) and from the uniformity of the limit $\|\widehat{\psi}_0(\cdot\pm\varepsilon\xi)-\widehat{\psi}_0(\cdot)\|\to 0$ as $\xi$ runs over any compact set. Finally, the removal of the $k\le K$ cutoff in the summation follows from the same majorization as (6.12) together with

$$\sum_{k>K}\frac{\big[2\mathcal{I}(a)\tau\big]^k}{k!}\le\frac{(C\tau)^K}{K!}\le (C\lambda^{\delta})^K\to 0\;.$$

For a fixed energy $e > 0$ we consider the continuous time Markov process $\{v(t)\}_{t\geq 0}$ on the energy surface $\Sigma_e$ with generator (1.12). This process is exponentially mixing with the uniform measure on $\Sigma_e$ being the unique invariant measure (see Lemma A.1 in the Appendix). Let $\mathcal{E}_e^\psi$ denote the expectation value with respect to this process starting from the initial state $\psi = \psi_0$ given by the normalized measure

$$\frac{|\widehat{\psi}(v)|^2\, \delta(e(v_1) - e)\, \mathrm{d}v}{\big[|\widehat{\psi}|^2\big](e)}$$

on $\Sigma_e$ (for the notations, see Section 1). Let $\mathcal{E}_e$ denote the expectation with respect to the equilibrium. Let $\mathrm{d}\mu_\psi(e) = \big[|\widehat{\psi}|^2\big](e)\mathrm{d}e$ be the energy distribution of $\psi$. The coarea formula,

$$\int_0^\infty \big[|\widehat{\psi}|^2\big](e)\, \mathrm{d}e = \int |\widehat{\psi}(v)|^2 \mathrm{d}v$$

and $\psi \in L^2$ guarantee that $\mathrm{d}\mu_\psi$ is absolutely continuous.

From (6.11) and $\tau = \lambda^{-\kappa} T$ we have

$$\sum_{k<K} W(k) = \int_0^\infty \mathcal{E}_e^\psi \mathcal{O}\big(\lambda^{\kappa/2} x(\tau), v(\tau)\big)\, \mathrm{d}\mu_\psi(e) + o(1) \quad \text{with} \quad x(\tau) := \int_0^\tau \frac{1}{2\pi} v(s) \mathrm{d}s\,.$$

Due to the exponential mixing and the continuity of $\mathcal{O}$, the replacement of $\mathcal{E}_\psi$ with the equilibrium measure $\mathcal{E}_e$ gives a negligible error since $\tau \to \infty$. By the central limit theorem for additive functionals of exponentially mixing Markov chains, $\lambda^{\kappa/2} x(\tau)$ converges to a centered Gaussian random variable with covariance matrix

$$\mathcal{E}_e\big[\lambda^{\kappa/2} x(\tau) \otimes \lambda^{\kappa/2} x(\tau)\big] = \frac{\lambda^\kappa}{(2\pi)^2} \iint_0^\tau \mathcal{E}_e\big[v(s) \otimes v(s')\big] \mathrm{d}s \mathrm{d}s' \to 2TD(e)\,.$$

Since the equilibrium measure is uniform, the covariance matrix is diagonal, $D_{ij}(e) = D_e\, \delta_{ij}$. The diffusion coefficient, $D_e$, is finite and positive. This proves Theorem 2.4. □

# A Mixing properties of the Boltzmann generator

The Boltzmann velocity process with generator $L_e$ introduced in (1.12) enjoys very good statistical properties. The proof uses standard arguments which we only indicate below.

**Lemma A.1** *For each $e > 0$ the Markov process $\{v(t)\}_{t\geq 0}$ with generator $L_e$ is uniformly exponentially mixing. The unique invariant measure is the uniform distribution, $[\,\cdot\,](e)/[1](e)$, on the energy surface $\Sigma_e$.*

*Sketch of the proof.* Let $\mathcal{P}^t(u, A)$ be the transition kernel for any $u \in \Sigma_e$, $A \subset \Sigma_e$. Since the transition rate $\sigma(u, v)$ is continuous on $\Sigma_e$ and $0 \in \text{supp}\,(\widehat{B})$ holds, there exists an open set $S \subset \mathbb{R}^d$, $\text{diam}(S) \leq \sqrt{2e}$ and there exists and $\delta = \delta(e) > 0$ such that $\sigma(u, v) \geq \delta$ whenever $u - v \in S$, $u, v \in \Sigma_e$. Since the state space $\Sigma_e$ is compact it follows that the transition kernel $\mathcal{P}^t$ satisfies a uniform Doeblin-type condition,

$$\inf_{u \in \Sigma_e} \int_0^1 \mathcal{P}^t(u, A) \, \mathrm{d}t \geq C(e)|A| \qquad A \subset \Sigma_e \; ,$$

with some $e$-dependent positive constant, where $|A|$ is the restriction of the Lebesgue measure (on $\Sigma_e$) of the set $A$. It is clear that the Markov process $\{v(t)\}_{t \geq 0}$ is irreducible and aperiodic, therefore it is uniformly exponentially mixing. Moreover, the rate of the mixing is uniform as $e$ runs through a compact energy interval since in this case $C(e)$ is uniformly separated away from zero. It is easy to see that the uniform measure on $\Sigma_e$ is invariant and by exponential mixing it is the only invariant measure. $\square$

# B Estimates on Propagators

## B.1 Proof of Lemma 2.1.

The following lemma proves (2.6) and (2.7). The proof of (2.8) is analogous but easier and will be omitted.

**Lemma B.1** *Let $\kappa < 1/6$ and $\eta$ satisfying $\lambda^{2+4\kappa} \leq \eta \leq \lambda^{2+\kappa}$. For any $0 \leq a \leq 1$ we have the following approximation result*

$$\int \left| \frac{1}{\alpha - \omega(p) + i\eta} - \frac{1}{\alpha - e(p) - \lambda^2 \Theta(\alpha) + i\eta} \right|^{2-a} |h(p-q)| \mathrm{d}p \tag{B.1}$$

$$\leq C\lambda^{1-6\kappa} \int \frac{|h(p-q)|}{|\alpha - e(p) - \lambda^2\Theta(\alpha) + i\eta|^{2-a}} \mathrm{d}p \; .$$

*Moreover, for any $0 \leq a < 1$ we have*

$$\int \frac{|h(p-q)| \, \mathrm{d}p}{|\alpha - e(p) - \lambda^2\Theta(\alpha) + i\eta|^{2-a}} \leq \frac{C_a \|h\|_{2d,0} \lambda^{-2(1-a)}}{\langle\alpha\rangle^{a/2} \langle |q| - \sqrt{2|\alpha|}\rangle} \tag{B.2}$$

*and*

$$\int \frac{|h(p-q)| \, \mathrm{d}p}{|\alpha - e(p) - \lambda^2\Theta(\alpha) + i\eta|} \leq \frac{C_a \|h\|_{2d,0} \left|\log \lambda\right| \, \log\langle\alpha\rangle}{\langle\alpha\rangle^{1/2} \langle |q| - \sqrt{2|\alpha|}\rangle} \; . \tag{B.3}$$

*Proof.* To prove (B.1), we rewrite it as

$$
\begin{aligned}
&\int \left| \frac{1}{\alpha - \omega(p) + i\eta} - \frac{1}{\alpha - e(p) - \lambda^2 \Theta(\alpha) + i\eta} \right|^{2-a} |h(p-q)| \, \mathrm{d}p \\
&= \int \left| \frac{\lambda^2 (\Theta(e(p)) - \Theta(\alpha))}{(\alpha - \omega(p) + i\eta)(\alpha - e(p) - \lambda^2 \Theta(\alpha) + i\eta)} \right|^{2-a} |h(p-q)| \, \mathrm{d}p \, .
\end{aligned} \tag{B.4}
$$

From the Hölder continuity,

$$
|\Theta(\alpha) - \Theta(\alpha')| \le C |\alpha - \alpha'|^{1/2} \tag{B.5}
$$

(Lemma 3.1 from [10]), we can bound (B.4) by

$$
\int \left| \frac{|e(p) - \alpha|^{1/2}}{(\alpha - \omega(p) + i\eta)} \frac{\lambda^2}{(\alpha - e(p) - \lambda^2 \Theta(\alpha) + i\eta)} \right|^{2-a} |h(p-q)| \mathrm{d}p \, .
$$

Since

$$
|e(p) - \alpha|^{1/2} \le |\omega(p) - \alpha|^{1/2} + O(\lambda) \, ,
$$

this integral is bounded by

$$
(\lambda^2 \eta^{-1/2} + \lambda^3 \eta^{-1})^{2-a} \int \frac{|h(p-q)| \, \mathrm{d}p}{|\alpha - e(p) - \lambda^2 \Theta(\alpha) + i\eta|^{2-a}} \, .
$$

Notice that in these estimates we used that the imaginary part of $\omega(p)$ is negative.

To prove the estimates (B.2) and (B.3), we rewrite the integrals by the co-area formula ($0 \le a \le 1$):

$$
\int \frac{|h(p-q)| \mathrm{d}p}{|\alpha - e(p) - \lambda^2 \Theta(\alpha) + i\eta|^{2-a}} = \int_0^\infty \frac{\mathrm{d}s}{|\alpha - s - \lambda^2 \Theta(\alpha) + i\eta|^{2-a}} \Xi(s) \tag{B.6}
$$

with

$$
\Xi(s) := \int_{|p| = \sqrt{2s}} \frac{|h(p-q)| \mathrm{d}p}{|\nabla e(p)|} = \frac{1}{\sqrt{2s}} \int_{|p| = \sqrt{2s}} |h(p-q)| \mathrm{d}p \, .
$$

Using the decay properties of $h$, we have

$$
\Xi(s) \le \frac{\|h\|_{2d,0}}{\langle |q| - \sqrt{2s} \rangle} \cdot \frac{\sqrt{s}}{\langle s \rangle} \, ,
$$

so

$$
\int \frac{|h(p-q)| \mathrm{d}p}{|\alpha - e(p) - \lambda^2 \Theta(\alpha) + i\eta|^{2-a}} \le C \|h\|_{2d,0} \int_0^\infty \frac{\sqrt{s} \, \mathrm{d}s}{\langle s \rangle \, \langle |q| - \sqrt{2s} \rangle |\alpha - s - \lambda^2 \Theta(\alpha) + i\eta|^{2-a}} \, .
$$

For $a=1$ the last integral can be directly estimated by $C\left|\log\eta\right|\langle\alpha\rangle^{-1/2}\langle|q|-\sqrt{2|\alpha|}\rangle\log\langle\alpha\rangle$, yielding (B.3).

To prove (B.2), i.e. for $a<1$, we recall that that $\Theta=\mathcal{R}-i\mathcal{I}$ with non-negative real $\mathcal{I}$ and $\mathcal{R}$. The last integral is estimated by

$$I:=\int_0^\infty\frac{\sqrt{s}\,\mathrm{d}s}{\langle s\rangle\;\langle|q|-\sqrt{2s}\rangle\;\left[(\widetilde{\alpha}-s)^2+(\lambda^2\mathcal{I}(\widetilde{\alpha})+\eta)^2\right]^{1-a/2}}$$

with $\widetilde{\alpha}:=\alpha-\lambda^2\mathcal{R}(\alpha)$. We used the Hölder continuity of $\mathcal{I}$, $\lambda^2\mathcal{I}(\alpha)=\lambda^2\mathcal{I}(\widetilde{\alpha})+O(\lambda^3)$ and the fact that the error can be absorbed into $\eta$.

First we assume that $|\widetilde{\alpha}|\leq 1$, then the estimate on $\Theta(\alpha)$ from Lemma 3.2. of [10] yields $\mathcal{I}(\widetilde{\alpha})\leq c_1|\widetilde{\alpha}|^{1/2}$. The $I$ integral can be estimated

$$I\leq\frac{1}{\langle|q|\rangle}\int_0^2\frac{\sqrt{s}\,\mathrm{d}s}{\left[(\widetilde{\alpha}-s)^2+(\lambda^2|\widetilde{\alpha}|^{1/2}+\eta)^2\right]^{1-a/2}}+\int_2^\infty\frac{\sqrt{s}\,\mathrm{d}s}{\langle s\rangle\;\langle|q|-\sqrt{2s}\rangle s^{2-a}}\;.$$

The second term is bounded by $\langle|q|\rangle^{-1}\sim\langle|q|-\sqrt{2|\alpha|}\rangle^{-1}$. In the first term we consider two cases. If $|\widetilde{\alpha}|\leq\widetilde{\beta}:=\lambda^2|\widetilde{\alpha}|^{1/2}+\eta$, then

$$\int_0^2\frac{\sqrt{s}\,\mathrm{d}s}{\left[(\widetilde{\alpha}-s)^2+(\lambda^2|\widetilde{\alpha}|^{1/2}+\eta)^2\right]^{1-a/2}}\leq\Big(\int_{|\widetilde{\alpha}-s|\leq 2\widetilde{\beta}}+\int_{|\widetilde{\alpha}-s|\geq 2\widetilde{\beta}}\Big)$$

$$\leq\widetilde{\beta}^{-(2-a)}\int_{|s|\leq 2\widetilde{\beta}}\sqrt{s}\mathrm{d}s+\int_{|\widetilde{\alpha}-s|\geq\widetilde{\beta}}\frac{\mathrm{d}s}{|\widetilde{\alpha}-s|^{3/2-a}}=O(\widetilde{\beta}^{a-1/2})\leq O(\lambda^{-2(1-a)})\;,$$

by using that in the second regime $s$ and $|s-\widetilde{\alpha}|$ are comparable and that $\widetilde{\beta}\geq\eta\geq\lambda^3$ in the last step. This proves (B.2) for $|\widetilde{\alpha}|\leq 1$.

Next we consider the regime $|\widetilde{\alpha}|\geq 1$, then $\mathcal{I}(\widetilde{\alpha})\leq c_1|\widetilde{\alpha}|^{-1/2}$ and we have

$$\begin{aligned}I\leq&\int\frac{\mathbf{1}(|\widetilde{\alpha}-s|\geq 1/2)\mathrm{d}s}{\langle s\rangle^{1/2}\langle|q|-\sqrt{2s}\rangle|\widetilde{\alpha}-s|^{2-a}}\\&+\int\frac{\mathbf{1}(|\widetilde{\alpha}-s|\leq 1/2)\mathrm{d}s}{\langle s\rangle^{1/2}\langle|q|-\sqrt{2s}\rangle\;\left[(\widetilde{\alpha}-s)^2+(\lambda^2|\widetilde{\alpha}|^{-1/2}+\eta)^2\right]^{1-a/2}}\;.\end{aligned}$$

The first integral is bounded by $C_a\langle\alpha\rangle^{-1/2}\langle|q|-\sqrt{2|\alpha|}\rangle^{-1}$, by using that $\langle\alpha\rangle\sim\langle\widetilde{\alpha}\rangle$. The second integral is bounded by

$$\frac{1}{\langle\widetilde{\alpha}\rangle^{1/2}\langle|q|-\sqrt{2\alpha}\rangle}\int_{-1/2}^{1/2}\frac{\mathrm{d}s}{\left[s^2+(\lambda^2|\widetilde{\alpha}|^{-1/2}+\eta)^2\right]^{1-a/2}}\leq\frac{C_a\lambda^{-2(1-a)}}{\langle\alpha\rangle^{a/2}\langle|q|-\sqrt{2\alpha}\rangle}$$

and this completes the proof of (B.2). $\square$

We now prove the more accurate estimate (2.9). We have

$$\frac{\lambda^2}{|\alpha - \overline{\omega}(p) - i\eta|^2} = \frac{\lambda^2}{\lambda^2 \mathcal{I}(e(p)) + \eta} \operatorname{Im} \frac{1}{\alpha - e(p) - \lambda^2 \mathcal{R}(e(p)) - i(\lambda^2 \mathcal{I}(e(p)) + \eta)} .$$

From the resolvent identity and with the notations $e = e(p)$, $\widetilde{\alpha} = \alpha - \lambda^2 \mathcal{R}(\alpha)$, the last term equals to $(I) + (II) + (III)$ with

$$\begin{aligned}
(I) :=& \frac{\lambda^2}{\lambda^2 \mathcal{I}(\widetilde{\alpha}) + \eta} \operatorname{Im} \frac{1}{\widetilde{\alpha} - e - i(\lambda^2 \mathcal{I}(\widetilde{\alpha}) + \eta)} \\
(II) :=& -\frac{\lambda^2}{\lambda^2 \mathcal{I}(\widetilde{\alpha}) + \eta} \frac{\lambda^2 (\mathcal{I}(e) - \mathcal{I}(\widetilde{\alpha}))}{\lambda^2 \mathcal{I}(e) + \eta} \operatorname{Im} \frac{1}{\alpha - e - \lambda^2 \Theta(e) - i\eta} \\
(III) :=& -\frac{\lambda^2}{\lambda^2 \mathcal{I}(\widetilde{\alpha}) + \eta} \operatorname{Im} \left[ \frac{1}{\widetilde{\alpha} - e - i(\lambda^2 \mathcal{I}(\alpha) + \eta)} \frac{\lambda^2 (\Theta(\alpha) - \Theta(e))}{\alpha - e - \lambda^2 \Theta(e) - i\eta} \right] .
\end{aligned} \tag{B.7}$$

Our goal is to estimate $\int |\widehat{B}(p-q)|^2 \big[(I) + (II) + (III)\big] \mathrm{d}p$.

We recall two continuity properties of $\Theta_\varepsilon(\alpha, r)$ from Lemma 3.1 of [10]:

$$|\Theta_\varepsilon(\alpha, r) - \Theta_e(\alpha, r')| \leq C \big| \, |r| - |r'| \, \big| \tag{B.8}$$

$$|\Theta_\varepsilon(\alpha, r) - \Theta_{\varepsilon'}(\alpha', r)| \leq C(|\varepsilon - \varepsilon'| + |\alpha - \alpha'|) \varepsilon^{-1/2} \tag{B.9}$$

if $\varepsilon \geq \varepsilon' > 0$. In estimating the integral of term (I), we first use (B.8) to change $q$ to $\widetilde{q}$ with $e(\widetilde{q}) = \widetilde{\alpha}$, then we use (B.9) with $\varepsilon' \to 0+0$ and $\varepsilon = \lambda^2 \mathcal{I}(\widetilde{\alpha}) + \eta = \mathcal{O}(\lambda^2)$ to obtain

$$\begin{aligned}
\int |\widehat{B}(p-q)|^2 (I) \, \mathrm{d}p &= \frac{\lambda^2}{\lambda^2 \mathcal{I}(\widetilde{\alpha}) + \eta} \Big[ \mathcal{I}(\widetilde{\alpha}) + \mathcal{O}(\lambda) + O(|\widetilde{\alpha} - e(q)|^{1/2}) \Big] \\
&\leq 1 + \mathcal{O}(\lambda^2 \eta^{-1} [\lambda + |\alpha - \omega(q)|^{1/2}]) .
\end{aligned}$$

In the second term of (B.7) we use (B.5) and we drop the positive $\mathcal{I}(\widetilde{\alpha})$ and $\mathcal{I}(e)$ terms in the denominators

$$\begin{aligned}
|(II)| &\leq C \Big( \frac{\lambda^2}{\eta} \Big)^2 \frac{|\lambda^2 \mathcal{I}(e(p)) + \eta| \; |\widetilde{\alpha} - e(p)|^{1/2}}{|\widetilde{\alpha} - e(p) + \lambda^2 (\mathcal{R}(\alpha) - \mathcal{R}(e(p)))|^2 + |\lambda^2 \mathcal{I}(e(p)) + \eta|^2} \\
&\leq C \lambda^2 \Big( \frac{\lambda^2}{\eta} \Big)^2 \frac{|\widetilde{\alpha} - e(p)|^{1/2}}{|\widetilde{\alpha} - e(p) + \lambda^2 (\mathcal{R}(\widetilde{\alpha}) - \mathcal{R}(e(p)))|^2 + \eta^2} ,
\end{aligned}$$

where we used that $\big[\lambda^2 (\mathcal{R}(\widetilde{\alpha}) - \mathcal{R}(\alpha))\big]^2 = \mathcal{O}(\lambda^6) \ll \eta^2$ based upon (B.5).

To perform the $\mathrm{d}p$ integration, we distinguish two regimes depending on whether $|\widetilde{\alpha} - e(p)|$ is bigger or smaller than $K\lambda^4$ for a sufficiently large fixed $K$. When $|\widetilde{\alpha} - e(p)| \geq K\lambda^4$, then $\lambda^2 |\mathcal{R}(\widetilde{\alpha}) - \mathcal{R}(e(p))| < \frac{1}{2} |\widetilde{\alpha} - e(p)|$, hence

$$|(II)| \leq C \lambda^2 \Big( \frac{\lambda^2}{\eta} \Big)^2 \frac{\eta^{-1/2}}{|\widetilde{\alpha} - e(p)| + \eta} ,$$

and, by using a bound analogous to (B.3), the corresponding integral is bounded by

$$C\lambda^6\eta^{-5/2}\int\frac{|\widehat{B}(p-q)|^2\,\mathrm{d}p}{|\widetilde{\alpha}-e(p)|+\eta}\le\mathcal{O}(\lambda^6\eta^{-5/2}|\log\eta|)\;.$$

When $|\widetilde{\alpha}-e(p)|\le K\lambda^4$, then we can trivially estimate $|(II)|\le C(\lambda^2\eta^{-1})^4$ and after the co-area formula, the volume factor is given by

$$\int_0^\infty \mathbf{1}(|\widetilde{\alpha}-s|\le K\lambda^4)\sqrt{s}S(s)\mathrm{d}s=O(\lambda^4)\;,$$

with

$$S(e):=\int_{S^{d-1}}|\widehat{B}(\sqrt{2e}(\phi_r-\phi))|^2\mathrm{d}\phi$$

where $\phi_r\in S^{d-1}$ is fixed. Recalling the properties of $S(e)$ from the proof of Lemma 3.2 in [10], we see that the contribution to the integral $\int|\widehat{B}(p-q)|^2(II)\,\mathrm{d}p$ is of order $\mathcal{O}((\lambda^3\eta^{-1})^4)$.

Finally, the last term in (B.7) is estimated as

$$|(III)|\le C\lambda^2\Big(\frac{\lambda^2}{\eta}\Big)\frac{1}{|\widetilde{\alpha}-e|+\eta}\;\frac{|\alpha-e|^{1/2}}{|\alpha-e+\lambda^2\mathcal{R}(e)|+\eta}\;.$$

In the regime where $|\alpha-e(p)|\ge K\lambda^2$ (with some large $K$) we obtain

$$|(III)|\le C\lambda^2\Big(\frac{\lambda^2}{\eta}\Big)\;\frac{|\alpha-e|^{1/2}}{(|\alpha-e|+\eta)^2}\le C\lambda^2\eta^{-1/2}\Big(\frac{\lambda^2}{\eta}\Big)\frac{1}{|\alpha-e|+\eta}$$

and after integration we collect $\mathcal{O}(\lambda^4\eta^{-3/2}|\log\eta|)$. In the regime where $|\alpha-e(p)|\le K\lambda^2$ we have $|(III)|\le\mathcal{O}(\lambda^5\eta^{-3})$ and the volume factor is $\mathcal{O}(\lambda^2)$, therefore the integral is $\mathcal{O}(\lambda^7\eta^{-3})$. Collecting the error terms we arrive at the proof of Lemma 2.1. □

## B.2 Proof of Lemma 6.1

We can assume that $f$ is a real function and write $f(p)=\langle p\rangle^{-2d}g(p)$ with $\|g\|_{2d,0}<C\|f\|_{4d,1}$. We can restrict the integration regime in (6.4) to $|p|\le\lambda^{-1}$ since the contribution of the outside regime is $O(\lambda^{2d})$ by a Schwarz inequality (to separate the two denominators) and a trivial application of Lemma B.1 with $a=0$. This large momentum cutoff will be done with the insertion of a function $\chi(\lambda\langle p\rangle)$ with a smooth, compactly supported $\chi$, $\chi\equiv 1$ on $[-1,1]$.

We can also assume that $|\alpha-e(p-r)|\le\lambda$, $|\beta-e(p+r)|\le\lambda$, otherwise at least one of the denominator can be estimated by $O(\lambda^{-1})$ and the other one integrated out by (2.6) to give $O(\lambda|\log\lambda|)$. Since $|\alpha-e(p-r)|\ge|\alpha-e(p)|-C(|p|+|r|)|r|\ge|\alpha-e(p)|-O(\lambda^{1+\kappa/4})$, we obtain that $|\alpha - e(p)| \le 2\lambda$ and similarly $|\beta - e(p)| \le 2\lambda$, in particular $|\alpha - \beta| \le 4\lambda$ and $|\gamma - e(p)| \le 2\lambda$.

We replace the first denominator of (6.4) by $\alpha - e(p) + p\cdot r - \lambda^2\bar\Theta(\gamma) - i\eta$. The error term of this replacement, by the resolvent expansion, is bounded by

$$\int \frac{\lambda^{9/2}\chi(\lambda\langle p\rangle)|f(p)|\mathrm{d}p}{|\alpha - \bar\omega(p-r) - i\eta|\;|\alpha - e(p) + p\cdot r - \lambda^2\bar\Theta(\gamma) - i\eta|\;|\beta - \omega(p+r) + i\eta|}\;, \tag{B.10}$$

where we have used the estimate

$$\alpha - e(p-r) - \lambda^2\overline{\theta}(p-r) - i\eta = \alpha - e(p) + p\cdot r - \lambda^2\bar\Theta(\gamma) - i\eta + O(\lambda^{5/2})\;,$$

which follows from the above restrictions on the integration domain $|r| \le \lambda^{2+\kappa/4}$ and the Hölder continuity (B.5). To estimate the error term (B.10), we bound the $\beta$ denominator trivially by $\eta^{-1}$ and use the Schwarz inequality to separate the remaining two denominators

$$\frac{1}{|\alpha - \bar\omega(p-r) - i\eta|\;|\alpha - e(p) + \dots|} \le \frac{1}{|\alpha - \bar\omega(p-r) - i\eta|^2} + \frac{1}{|\alpha - e(p) + \dots|^2}\;.$$

The integral of the first term can be bounded by Lemma B.1 with $a = 0$; for the second term we rewrite $e(p) + p\cdot r = e(p+r) - \frac{1}{2}r^2$ and use (2.8) after a shift in $\alpha$ and $p$. We arrive at $\Omega = \Omega_0 + O(\lambda^{1/2-4\kappa})$ with

$$\Omega_0 = \int \frac{\lambda^2\chi(\lambda\langle p\rangle)f(p)\mathrm{d}p}{\Big(\alpha - e(p) + p\cdot r - \lambda^2\bar\Theta(\gamma) - i\eta\Big)\Big(\beta - e(p) - p\cdot r - \lambda^2\Theta(\gamma) + i\eta\Big)}\;.$$

To compute $\Omega_0$ we can choose a coordinate system where the vector $r$ points in the $n$-th direction: $r = |r|(0,\dots,0,1)$. We can write

$$\Omega_0 = \lambda^2\int_{\mathbb{R}} \frac{\mathrm{d}p_\|}{\alpha - \beta + 2|r|\,p_\| - 2i[\lambda^2\mathcal{I}(\gamma) + \eta]} \tag{B.11}$$

$$\times\int_{\mathbb{R}^{n-1}}\Big[\frac{1}{\beta - e(p) - p_\||r| - \lambda^2\Theta(\gamma) + i\eta} - \frac{1}{\alpha - e(p) + p_\|\,|r| - \lambda^2\bar\Theta(\gamma) - i\eta}\Big]\chi(\lambda\langle p\rangle)f(p)\mathrm{d}p_\perp\;.$$

**Lemma B.2** *Let $F$ be a $C^1$-function on $\mathbb{R}$ with $|F(Q)| \le C\langle Q\rangle^{-2}$ and let*

$$Y(z) := \int_0^\infty \frac{F(Q)}{z - Q}\,\mathrm{d}Q$$

*for any $z = \alpha + i\varepsilon$ with $0 < \varepsilon \le 1/2$. Then*

$$|Y(z) - Y(z')| \le C|F(0)|\;|\log z - \log z'| + |z - z'||\log\varepsilon|\|F\|_{2d,1}\;, \tag{B.12}$$

*where $z' = \alpha' + i\varepsilon'$ and $\varepsilon \ge \varepsilon' > 0$.*

*Proof.* This lemma is essentially Lemma 3.10 in [7]. For completeness we recall the proof. Choose a branch of the complex logarithm on the upper half plane and use integration by parts:

$$|Y(z) - Y(z')| \le |F(0)||\log z - \log z'| + \Big| \int_0^\infty F'(Q)\big[\log(z-Q) - \log(z'-Q)\big]\mathrm{d}Q\Big| \,.$$

The second term is estimated by

$$\int_{\Gamma(z,z')} \mathrm{d}|\xi| \int_0^\infty \frac{|F'(Q)|}{|\xi - Q|}\mathrm{d}Q \;,$$

where $\Gamma(z,z')$ is any path in the upper half plane that connects $z$ and $z'$ and $\mathrm{d}|\xi|$ is the arclength measure. A simple exercise shows

$$\int_0^\infty \frac{|F'(Q)|}{|\xi - Q|}\,\mathrm{d}Q \le C\|F\|_{2d,1}|\log(\mathrm{Im}\,\xi)| \;.$$

Choose a path from $z = \alpha + i\varepsilon$ to $\alpha' + i\varepsilon$ then to $\alpha' + i\varepsilon'$ along straight line segments. After integration we obtain (B.12). $\square$

We now change the denominators in the square bracket in (B.11) to $\gamma - e(p) \pm i\eta$. This requires a change of order $O(\lambda)$ in the denominators using the estimates on $|\alpha - \gamma|$ and $p_\| |r|$. With the help of Lemma B.2 such change yields an error of order $\lambda^3|\log\lambda|\eta^{-1}\|f\|_{2d,1}$ in $\Omega_0$. After these changes, we can remove the cutoff $\chi(\lambda\langle p\rangle)$ at a price of $O(\lambda^{2d})$ as before, and we have

$$\Omega_0 = \lambda^2\int_{\mathbb{R}} \frac{\mathrm{d}p_\|}{\alpha - \beta + 2|r|p_\| - 2i[\lambda^2\mathcal{I}(\gamma) + \eta]}\int_{\mathbb{R}^{d-1}} \Big[\frac{1}{\gamma - e(p) + i\eta} - \frac{1}{\gamma - e(p) - i\eta}\Big] f(p)\mathrm{d}p_\perp \tag{B.13}$$

modulo negligible errors involving $\|f\|_{4d,1}$.

The inner integral is evaluated as

$$I := i\,\mathrm{Im}\int_0^\infty \frac{f^*(u,p_\|)u^{\frac{d-3}{2}}\mathrm{d}u}{\gamma - \frac{1}{2}p_\|^2 - \frac{1}{2}u + i\eta} \qquad \text{with} \quad f^*(u,p_\|) := \int_{S^{d-2}} f(u^{1/2}\theta, p_\|)\mathrm{d}\theta \;.$$

For $p_\|$ in the range $\gamma - \frac{1}{2}p_\|^2 \le -\lambda$, we have

$$|I| \le \frac{C\eta\|f\|_{4d,0}}{\langle p_\|\rangle^{2d}}\int_0^\infty \frac{u^{\frac{d-3}{2}}\mathrm{d}u}{|\lambda + \frac{1}{2}u|^2\,\langle u\rangle^{2d}} = O(\eta\lambda^{-1}) \;,$$

by using the decay of $f^*$ inherited from $f$. The contribution of this regime to $\Omega_0$ is therefore of order $\lambda|\log\lambda|$, and hence negligible. In the regime $|\gamma - \frac{1}{2}p_\|^2| \le \lambda$ one can estimate $|I| \leq C\langle p_\| \rangle^{-2d}$. The $\mathrm{d}p_\|$-volume is at most $O(\lambda^{1/2})$, so the contribution of this regime to $\Omega_0$ is at most of order $\lambda^{5/2}\eta^{-1}$, and hence also negligible.

Finally, we can concentrate on the regime $\gamma - \frac{1}{2}p_\|^2 \geq \lambda$. We can use the estimate (for $\varepsilon > \varepsilon' > 0$)

$$\mathrm{Im} \int_{-\varepsilon}^{\infty} \frac{g(x)}{x + i\varepsilon'}\, \mathrm{d}x = -\pi g(0) + O(\varepsilon'/\varepsilon) + O(\varepsilon' |\log \varepsilon'|)$$

if $g \in C^1$ with a bounded derivative. We obtain, with $\varepsilon = \gamma - \frac{1}{2}p_\|^2$, $\varepsilon' = \eta$, that

$$I = -2\pi i (2\gamma - p_\|^2)^{\frac{d-3}{2}} f^*(2\gamma - p_\|^2, p_\|) + O(\lambda^{1+4\kappa}) \ ,$$

where the error is integrable in $p_\|$. Therefore it is negligible in $\Omega_0$. Substituting the main term into (B.13), we obtain the main term in (6.4). $\square$.

# C General estimates on circle graphs

We define four operations on a partition given on the vertex set of a circle graph on $N$ vertices and we estimate how the $E$-value of the partition changes. Operation I was already defined in Section 9 of [10], here we repeat the definition and the corresponding estimate for convenience.

**Operation I: Breaking up lumps**

Consider a Feynman graph on $N$ vertices (Section 4.1). Given a partition of the set $\mathcal{V} \setminus \{0, 0^*\}$, $\mathbf{P} = \{P_\mu \ : \ \mu \in I(\mathbf{P})\} \in \mathcal{P}_\mathcal{V}$, we define a new partition $\mathbf{P}^*$ by breaking up one of the lumps into two smaller nonempty lumps. Let $P_\nu = P_{\nu'} \cup P_{\nu''}$ with $P_{\nu'} \cap P_{\nu''} = \emptyset$ and $\mathbf{P}^* = \{P_{\nu'}, P_{\nu''}, P_\mu \ : \ \mu \in I(\mathbf{P}) \setminus \{\nu\}\}$. In particular $I(\mathbf{P}^*) = I(\mathbf{P}) \cup \{\nu', \nu''\} \setminus \{\nu\}$ and $m(\mathbf{P}^*) = m(\mathbf{P}) + 1$. The following estimate was proven in Lemma 9.5 of [10].

**Lemma C.1** *With the notation above, we have*

$$E_{(*)g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha}) \leq \int_{|r| \leq N\zeta} \mathrm{d}r \ E_{(*)g}(\mathbf{P}^*, \mathbf{u}^*(r, \nu), \boldsymbol{\alpha}) \ ,$$

*where the new set of momenta* $\mathbf{u}^* = \mathbf{u}^*(r, \nu)$ *is given by* $u^*_\mu := u_\mu$, $\mu \in I(\mathbf{P}) \setminus \{\nu\}$ *and* $u^*_{\nu'} = u_\nu - r$, $u^*_{\nu''} = r$. *In our estimates we will always have* $N \leq 2K$ *and then*

$$\sup_{\mathbf{u}} E_{(*)g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha}) \leq \Lambda E_{(*)g} \sup_{\mathbf{u}} (\mathbf{P}^*, \mathbf{u}, \boldsymbol{\alpha})$$

*with* $\Lambda := [CK\zeta]^d = O(\lambda^{-2d\kappa - O(\delta)})$ *(see* (2.18) *and* (4.8)*).*

**Operation II: Removing the lump of a single vertex**

Let $v \in \mathcal{V} \setminus \{0, 0^*\}$ be a vertex and let $\mathbf{P} \in \mathcal{P}_{\mathcal{V}}$ such that $P_\sigma = \{v\}$ for some $\sigma \in I(\mathbf{P})$, i.e. the single element set $\{v\}$ is a lump. Define $\mathcal{V}^* := \mathcal{V} \setminus \{v\}$, $\mathcal{L}(\mathcal{V}^*) := \mathcal{L}(\mathcal{V}) \cup \{(v-1, v+1)\} \setminus \{(v-1, v), (v, v+1)\}$, i.e. we simply remove the vertex $v$ from the circle graph and connect the vertices $v-1, v+1$. Let $\mathbf{P}^* \in \mathcal{P}_{\mathcal{V}^*}$, $\mathbf{P}^* := \mathbf{P} \setminus \{\ \{v\}\ \}$ be $\mathbf{P}$ after simply removing the lump $\{v\}$. In particular, $I(\mathbf{P}^*) = I(\mathbf{P}) \setminus \{\sigma\}$.

**Lemma C.2** *With the notations above*

$$\sup_{\mathbf{u}} E_{(*)g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha}) \leq C\lambda\eta^{-1} \sup_{\mathbf{u}^*} E_{(*)g+1}(\mathbf{P}^*, \mathbf{u}^*, \boldsymbol{\alpha}) \ . \tag{C.1}$$

*If both neighbors of* $0^*$, $v \neq v'$, *form single lumps in* $\mathbf{P}$, *then both of these lumps can be simultaneously removed to obtain a partition* $\mathbf{P}^* := \mathbf{P} \setminus \{\{v\}, \{v'\}\}$ *with the estimate*

$$\sup_{\mathbf{u}} E_{*g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha}) \leq C\lambda^2 \ \sup_{\mathbf{u}^*} E_{g+2}(\mathbf{P}^*, \mathbf{u}^*, \boldsymbol{\alpha}) \ . \tag{C.2}$$

*Proof.* The factor $\lambda$ in the estimate (C.1) is due to the fact that each vertex (apart from 0 and $0^*$) carries a factor $\lambda$ and $|\mathcal{V}^*| = |\mathcal{V}| - 1$. Let $P_\nu$ be the lump of the vertex $v-1$ right before to $v$ in the circular ordering and assume $v - 1 \neq 0, 0^*$ (otherwise we consider $v+1$ and the proof is slightly modified). We use the trivial bound

$$\frac{1}{|\alpha_{e_{v-}} - \omega(w_{e_{v-}}) + i\eta|} \leq \eta^{-1} \tag{C.3}$$

(recall that $\operatorname{Im} \omega \leq 0$ from Lemma 3.2 of [10]) and the bound

$$|\widehat{B}(w_{e_{v+}} - w_{e_{v-}})| \ |\widehat{B}(w_{e_{v-}} - w_{e_{(v-1)-}})| \leq \frac{C}{\langle w_{e_{v+}} - w_{e_{(v-1)-}} \rangle^{2d}} \tag{C.4}$$

(uniformly in $w_{e_{v-}}$) to obtain the necessary decay between the two newly consecutive momenta. The same bound holds if some of the $\widehat{B}(\cdot)$ on the left hand side is replaced with $\langle \cdot \rangle^{-2d}$ due to the set $\mathcal{G}$. Now we integrate $w_{e_{v-}}$ to obtain a new delta function from

$$\int \mathrm{d}\mu(w_{e_{v-}}) \delta\Big(w_{e_{v+}} - w_{e_{v-}} - u_\sigma\Big) \delta\Big(w_{e_{v-}} + \sum_{e \in L_\pm(P_\nu)\,:\, e \neq e_{v-}} \pm w_e - u_\nu\Big)$$

$$\leq \delta\Big(w_{e_{v+}} + \sum_{e \in L_\pm(P_\nu)\,:\, e \neq e_{v-}} \pm w_e - (u_\sigma + u_\nu)\Big)$$

and clearly

$$w_{e_{v+}} + \sum_{e \in L_\pm(P_\nu)\,:\, e \neq e_{v-}} \pm w_e = \sum_{e \in L_\pm(P_\nu^*)} \pm w_e \ .$$

The new auxiliary momentum associated to $P_\nu$ is $u_\nu + u_\sigma$ and $P_\sigma$ disappeared, so the sum of the auxiliary momenta remain unchanged, and (4.5) continues to hold. This proves (C.1).

For the proof of (C.2), if $v$ and $v'$ are the vertices on both sides of $0^*$, then they can be removed and their neighbours can be connected directly to $0^*$ yielding a non-truncated value of a graph with two vertices less. This gives a factor $\lambda^2$. The appropriate redefinition of the auxiliary momenta is straightforward. $\square$

**Operation III: Removing half of a gate**

Let $v, v+1 \in \mathcal{V} \setminus \{0, 0^*\}$ two subsequent vertices and let $\mathbf{P} \in \mathcal{P}_\mathcal{V}$ such that $v \equiv v+1 \pmod{\mathbf{P}}$. In the main application this will arise when $v, v+1$ are connected and form a gate. Define $\mathcal{V}^* := \mathcal{V} \setminus \{v+1\}$, $\mathcal{L}(\mathcal{V}^*) := \mathcal{L}(\mathcal{V}) \cup \{(v, v+2)\} \setminus \{(v, v+1), (v+1, v+2)\}$, i.e. we simply remove the vertex $v+1$ from the circle graph with the adjacent edges and add a new edge between the vertices $v, v+2$. Let $\mathbf{P}^* \in \mathcal{P}_{\mathcal{V}^*}$ be identical to the partition $\mathbf{P}$ except that $v+1$ is simply removed from its lump. In particular, $I(\mathbf{P}) = I(\mathbf{P}^*)$.

**Lemma C.3** *With the notations above*

$$E_{(*)g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha}) \leq C\lambda |\log \eta| \; E_{(*)g+1}(\mathbf{P}^*, \mathbf{u}, \boldsymbol{\alpha}) \; .$$

*Proof.* Note that the momentum $w_{e_{v+}}$ of the edge between $v$ and $v+1$ does not appear in the delta functions in the definition of $E_{(*)g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha})$ (see (4.9)). Before integrating out this momentum in (4.9), we use the bound

$$|\widehat{B}(w_{e_{v+}} - w_{e_{v-}})| \, |\widehat{B}(w_{e_{v-}} - w_{e_{(v-1)-}})|$$

$$\leq \frac{C}{\langle w_{e_{v+}} - w_{e_{(v-1)-}} \rangle^{2d}} \Big[ \frac{1}{\langle w_{e_{v+}} - w_{e_{v-}} \rangle^{2d}} + \frac{1}{\langle w_{e_{v-}} - w_{e_{(v-1)-}} \rangle^{2d}} \Big]$$

to ensure the decay between the momenta $w_{e_{v-}}$ and $w_{e_{(v+2)-}}$, that are consecutive in the new graph. The same bound holds if some of the $\widehat{B}(\cdot)$ is already replaced with $\langle \cdot \rangle^{-2d}$. The integration of $w_{e_{v-}}$ yields $C|\log \eta|$ by using (2.6). $\square$

**Operation IV: Removing an isolated gate**

Let $v, v+1 \in \mathcal{V} \setminus \{0, 0^*\}$ be two subsequent vertices and let a partition $\mathbf{P} \in \mathcal{P}_\mathcal{V}$ such that $v \equiv v+1 \pmod{\mathbf{P}}$. Define $\mathcal{V}^* := \mathcal{V} \setminus \{v, v+1\}$, $\mathcal{L}(\mathcal{V}^*) := \mathcal{L}(\mathcal{V}) \cup \{(v-1, v+2)\} \setminus \{(v-1, v), (v, v+1), (v+1, v+2)\}$, i.e. we simply remove the gate. Let $\mathbf{P}^* \in \mathcal{P}_{\mathcal{V}^*}$ be $\mathbf{P}$ after removing the lump $\{v, v+1\}$. Combining Operations III and II, we immediately obtain:

**Lemma C.4** *With the notations above*

$$\sup_{\mathbf{u}} E_{(*)g}(\mathbf{P}, \mathbf{u}, \boldsymbol{\alpha}) \leq C\lambda^2\eta^{-1}|\log \eta| \sup_{\mathbf{u}^*} E_{(*)g+2}(\mathbf{P}^*, \mathbf{u}^*, \boldsymbol{\alpha}) . \qquad \square$$

Note that this bound is not optimal. The removal of a gate affects the value of the graph only a by constant factor, but the corresponding estimate is more complicated and we do not aim at optimizing the value of $\kappa$.

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